

2.4 The Forces Causing the Motion of Water in a Porous Media

We first use the control volume approach to consider the applied forces acting on a small volume of fluid within a porous medium. These forces, which are depicted in **Figure 2.7**, are the weight of the fluid and the forces due to fluid pressure. When the fluid is in motion, there are also frictional forces that tend to oppose the motion.

First we have to review some important definitions and notations,

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \Rightarrow \text{Force} = \text{Pressure} \times \text{Area} \quad (2.15)$$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} \Rightarrow \text{Mass} = \text{Density} \times \text{Volume} \quad (2.16)$$

$$\frac{\partial \xi}{\partial t}, \frac{d\xi}{dt}$$

time derivation of ξ

$$\nabla \xi = \frac{\partial \xi}{\partial x} i + \frac{\partial \xi}{\partial y} j + \frac{\partial \xi}{\partial z} k$$

gradient of scalar function ξ

$$\nabla \cdot \xi$$

divergence of vector ξ

If $\xi = u(x, y, z)i + v(x, y, z)j + \omega(x, y, z)k$ then, divergence $\xi = \nabla \cdot \xi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z}$

$$\frac{\partial \rho}{\partial T}$$

$$[M / (^{\circ}CL^3)]$$

a constant value of density change with temperature

$$\frac{\partial \rho}{\partial C}$$

$$[M_s / L^3]$$

a constant value of density change with concentration

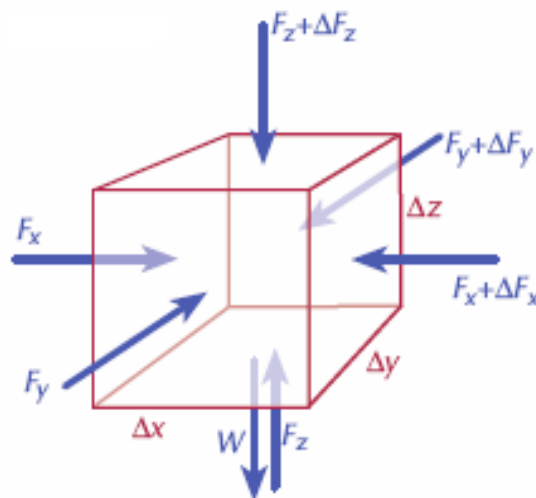


Figure 2.7 The applied forces acting on a small volume of fluid within a porous media

The resultant of the fluid weight and the forces due to hydrostatic pressure is given by:

$$\mathbf{F} = (F_x - (F_x + \Delta F_x))\mathbf{i} + (F_y - (F_y + \Delta F_y))\mathbf{j} + (F_z - (F_z + \Delta F_z) - W)\mathbf{k} \quad (2.17)$$

where \mathbf{i} , \mathbf{j} and \mathbf{k} are the unit vectors in the x , y and z directions respectively.

$$\text{But, Weight} = \text{mass} \times \text{gravity acceleration} = \text{Volume} \times \rho \times g = \Delta x \Delta y \Delta z \times \rho \times g$$

In terms of the pressures on each face of the volume, the resultant force can be written:

$$\mathbf{F} = (P_x - (P_x + \Delta P_x))\Delta_y \Delta_z \mathbf{i} + (P_y - (P_y + \Delta P_y))\Delta_z \Delta_x \mathbf{j} + (P_z - (P_z + \Delta P_z))\Delta_x \Delta_y \mathbf{k} - (\rho g \Delta_x \Delta_y \Delta_z)\mathbf{k} \quad (2.18)$$

or,

$$\mathbf{F} = -\Delta P_x \Delta_y \Delta_z \mathbf{i} - \Delta P_y \Delta_z \Delta_x \mathbf{j} - (\Delta P_z + \rho g \Delta z)\Delta_x \Delta_y \mathbf{k} \quad (2.19)$$

Dividing by $\Delta_x \Delta_y \Delta_z$ gives the force per unit volume

$$F = -\frac{\Delta P_x}{\Delta_x} \mathbf{i} - \frac{\Delta P_y}{\Delta_y} \mathbf{j} - \left(\frac{\Delta P_z}{\Delta_z} + \rho g \right) \mathbf{k} \quad (2.20)$$

Now letting Δ_x , Δ_y and Δ_z tend to zero gives,

$$\begin{aligned} F &= -\frac{\partial P}{\partial x} \mathbf{i} - \frac{\partial P}{\partial y} \mathbf{j} - \left(\frac{\partial P}{\partial z} + \rho g \right) \mathbf{k} \\ F &= -\frac{\partial P}{\partial x} \mathbf{i} - \frac{\partial P}{\partial y} \mathbf{j} - \frac{\partial P}{\partial z} \mathbf{k} - \rho g \mathbf{k} \\ &= -\nabla P - \rho g \mathbf{k} \\ &= -\nabla P - \rho g \nabla z \end{aligned} \quad (2.21)$$

Notice here that the expressions ∇z and \mathbf{k} are identical.

Following Darcy, we now apply the result that the specific discharge is proportional to the applied force. Initially we consider the cases of isotropic and anisotropic media separately.

2.4.1 Flow in Isotropic Porous Media (The General Case)

In an isotropic medium we assume that there are no preferential directions within the medium and consequently that the specific discharge is in the same direction as the resultant force on a volume of fluid. Darcy's Law states that the specific discharge is proportional to the head or energy gradient which represents the force applied to a volume of water and so this can be written.

$$\begin{aligned}
 q &= -\frac{k}{\mu}(\nabla P + \rho g \nabla z) \\
 &= -\frac{k}{\mu} \rho g \left(\frac{1}{\rho g} \nabla P + \nabla z \right)
 \end{aligned}
 \tag{2.22}$$

The constant of proportionality has been decomposed into the ratio of two quantities:

- ✓ k the intrinsic permeability of the medium $[L^2]$
- ✓ μ the dynamic viscosity of the fluid $[ML^{-1}T^{-1}]$

Notice that the intrinsic permeability is a property of the porous medium whereas the dynamic viscosity is a property of the fluid.

Notice also that the expression for \mathbf{q} above **cannot** be simplified to:

$$q = -\frac{k}{\mu} \rho g \nabla \left(\frac{P}{\rho g} + z \right)
 \tag{2.23}$$

Since $1/\rho g$ is outside the ∇ operator and then it requires that \mathbf{p} and \mathbf{g} be constant.

2.4.2 Flow in Isotropic Porous Media (Constant Density)

The acceleration due to gravity is a physical constant, but if it is assumed that \mathbf{p} is also constant, then we can write:

$$\begin{aligned}
 q &= -\frac{k}{\mu} \rho g \nabla \left(\frac{P}{\rho g} + z \right) \\
 &= -\frac{k}{\mu} \rho g \nabla h \\
 &= -K \nabla h
 \end{aligned}
 \tag{2.24}$$

or, it can also be written in different notation as:

$$q = -K \left(\frac{\partial h}{\partial x} i + \frac{\partial h}{\partial y} j + \frac{\partial h}{\partial z} k \right)
 \tag{2.25}$$

where K is known as the Hydraulic Conductivity $[LT^{-1}]$. Notice here that the hydraulic conductivity is given by:

$$K = \frac{k}{\mu} \rho g
 \tag{2.26}$$

and is a property both of the porous medium, (since it is a function of k), and of the fluid contained in it, since it depends also on μ . **Equation 2.25** should be recognized as a simple extension of Darcy's experimental results to two or three dimensions. However, it is important to realize that **equation 2.25** is valid only if the density of the fluid is constant.

Thus, if the density of the groundwater is affected significantly by, say, the presence of dissolved salts, as in the case of seawater intrusion problems, then the use of this equation is inappropriate and the problem must be formulated in terms of fluid pressures rather than piezometric head **equation 2.25**.

2.4.3 Flow in Anisotropic Porous Media (General Case)

The form of Darcy's Law given in **equation 2.22** implies that the specific discharge vector is in the same direction as the applied force. In other words pushing groundwater in one direction results in flow in that and only that direction. In many real cases this is not true. Consider, for example, the case of flow in fractured rock. It may be that fractures in a particular orientation are dominant in terms of conducting water, in which case there will be a tendency for water to flow in that direction even if the head gradient is not aligned with it. A similar phenomenon occurs in the vertical dimension in stratified aquifers, e.g. alluvial aquifers composed of sands and gravels (medium to high permeability) interspersed with extensive layers of clay (low permeability).

In an isotropic medium the component of the applied force along each coordinate axis produces flow in that direction only. However, in anisotropic medium each component of the applied force will potentially produce a specific discharge in each of the three coordinate directions. So, for example, the specific discharge in the direction x is caused in part by all the components of the applied force \mathbf{F} . That is

$$q_x = -\frac{k_{xx}}{\mu} \frac{\partial P}{\partial x} - \frac{k_{xy}}{\mu} \frac{\partial P}{\partial y} - \frac{k_{xz}}{\mu} \left(\frac{\partial P}{\partial z} + \rho g \right) \quad (2.27)$$

For some constants k_{xx} , k_{xy} and k_{xz}

In this case we express Darcy's Law in full as

$$\begin{aligned} q_x &= -\frac{1}{\mu} \left[k_{xx} \frac{\partial P}{\partial x} + k_{xy} \frac{\partial P}{\partial y} + k_{xz} \left(\frac{\partial P}{\partial z} + \rho g \right) \right] \\ q_y &= -\frac{1}{\mu} \left[k_{yx} \frac{\partial P}{\partial x} + k_{yy} \frac{\partial P}{\partial y} + k_{yz} \left(\frac{\partial P}{\partial z} + \rho g \right) \right] \\ q_z &= -\frac{1}{\mu} \left[k_{zx} \frac{\partial P}{\partial x} + k_{zy} \frac{\partial P}{\partial y} + k_{zz} \left(\frac{\partial P}{\partial z} + \rho g \right) \right] \end{aligned} \quad (2.28)$$

which is more suitably expressed in matrix form

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = -\frac{1}{\mu} \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} + \rho g \end{bmatrix} \quad (2.29)$$

or as

$$q = -\frac{k}{\mu} [\nabla P + \rho g \nabla_z] \quad (2.30)$$

2.4.4 Flow in Anisotropic Porous Media (Constant Density)

In the case of a homogeneous fluid, the density is constant and we can write

$$q = -K \nabla h \quad (2.31)$$

and if the principal axes of permeability are aligned with the x , y and z axes then the hydraulic conductivity tensor is given by

$$K = \begin{bmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix} \quad (2.32)$$

2.4.5 Change of Coordinates

It is natural to ask whether the three principal coefficients can be used in some way if the principal axes of the permeability are not aligned with the x , y and z axes. The following argument shows how the full permeability tensor can be determined from the principal components. The argument is developed in the two-dimensional case with fluid of constant density, but the result extends easily to three dimensions and is essentially similar in form for the variable density case.

Let the coordinate axes be x and y and let the principal axes of permeability be X and Y as shown in **Figure 2.8**.

Then,

- X the direction along which the permeability has an absolute maximum value.
- Y a perpendicular direction to X , in which permeability has an absolute minimum value.

(X, Y) : principle directions of permeability

$k_{\max}(x, y)$	[L/T]	absolute maximum value of permeability
$k_{\min}(x, y)$	[L/T]	absolute minimum value of permeability
$\theta(x, y)$	[degree]	angle from $+x$ -coordinate axis to direction of maximum permeability

Then the components of specific discharge are given by

$$\begin{aligned} q_x &= q_X \cos \theta - q_Y \sin \theta \\ q_y &= q_X \sin \theta + q_Y \cos \theta \end{aligned} \quad (2.33)$$

From **Figure 2.8** it can be seen that the following relationships hold between the two coordinate systems.

$$\begin{aligned} x &= X \cos \theta & y &= -Y \sin \theta \\ y &= X \sin \theta & x &= Y \cos \theta \end{aligned} \quad (2.34)$$

from which we can derive the partial derivatives of h with respect to X and Y in terms of x, y and θ as follows

$$\begin{aligned}
\frac{\partial h}{\partial X} &= \frac{\partial h}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial h}{\partial y} \frac{\partial y}{\partial X} \\
&= \frac{\partial h}{\partial x} \cos \theta + \frac{\partial h}{\partial y} \sin \theta \\
\frac{\partial h}{\partial Y} &= \frac{\partial h}{\partial x} \frac{\partial x}{\partial Y} + \frac{\partial h}{\partial y} \frac{\partial y}{\partial Y} \\
&= -\frac{\partial h}{\partial x} \sin \theta + \frac{\partial h}{\partial y} \cos \theta
\end{aligned}
\tag{2.35}$$

Substituting **equation 2.35** into Darcy's Law (**equation 2.31**), and the resulting equation into **equation 2.33** gives

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = - \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix}
\tag{2.36}$$

but, From the Permeability Ellipse shown in **Figure 2.8**,

$$\begin{aligned}
\text{semi-major axes} &: \sqrt{K_{\max}} \\
\text{semi-minor axes} &: \sqrt{K_{\min}} \\
\text{the length of any radius} &= \sqrt{K}
\end{aligned}$$

So;

$$\begin{aligned}
K_{xx} &= K_{\max} \cos^2 \theta + K_{\min} \sin^2 \theta \\
K_{xy} = K_{yx} &= (K_{\max} - K_{\min}) \sin \theta \cos \theta \\
K_{yy} &= K_{\max} \sin^2 \theta + K_{\min} \cos^2 \theta
\end{aligned}
\tag{2.37}$$

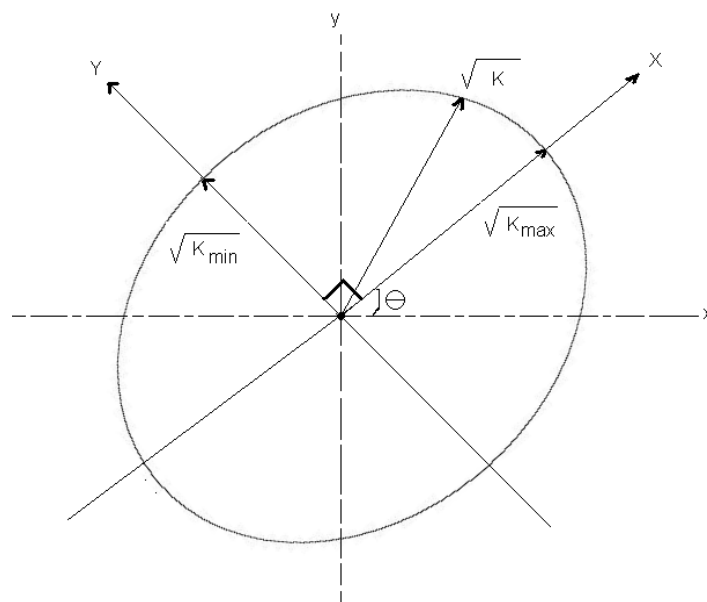


Figure 2.8

2.5 The Continuity Equation for Fluid Flow in Porous Media

The principal of the conservation of mass is fundamental to all hydrology and is expressed mathematically by the **continuity equation**.

The development of the equation presented in this section is perfectly general. The equation is developed by considering the change in mass of a substance stored in a small control volume over a short time period.

The fundamental principle is that, for a fixed volume over a given period of time:

$$\text{Increase in mass stored} = \text{mass inflow} - \text{mass outflow}$$

The continuity equation expresses the conservation of fluid mass within the porous medium. It can be developed by considering the mass balance within a small volume of aquifer shown in **Figure 2.9**. At this stage we assume that the flow through the control box is due entirely to flow across the faces of the box, i.e. that there are no fluid sources or sinks within the box itself. We also allow the porosity of the aquifer and the fluid density to vary in time.

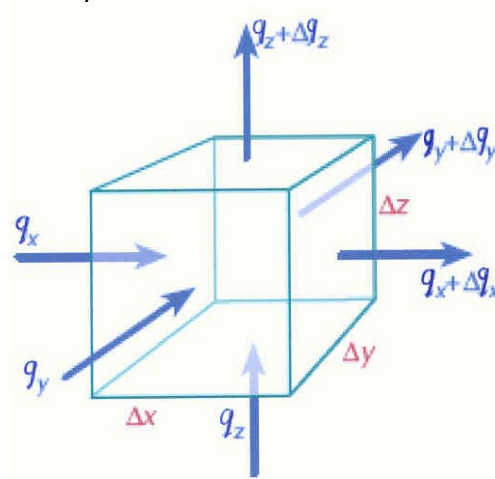


Figure 2.9 The mass fluxes across the walls of the control volume

Let n [dimensionless] be the porosity of the volume at time t , M [M] the mass of fluid contained in the volume at time t , and ρ [ML⁻³] the density of the fluid at time t .

Then the change in mass in the volume during time Δt can be determined by calculating the difference in the mass at the end and the beginning of the time interval. This can be done in two ways.

First, by calculating the mass by multiplying the density of the fluid with the porosity of the aquifer multiplied by the volume of the control box. This gives

$$\Delta M = [(\rho + \Delta\rho)(n + \Delta n) - \rho n] \Delta x \Delta y \Delta z \quad (2.38)$$

Second, by calculating the mass in terms of the volume fluxes (specific discharges) in and out of the control box, the change in mass is given by:

$$\begin{aligned} \Delta M &= q_x \rho \Delta y \Delta z \Delta t - (q_x + \Delta q_x)(\rho + \Delta\rho) \Delta y \Delta z \Delta t \\ &+ q_y \rho \Delta z \Delta x \Delta t - (q_y + \Delta q_y)(\rho + \Delta\rho) \Delta z \Delta x \Delta t \\ &+ q_z \rho \Delta x \Delta y \Delta t - (q_z + \Delta q_z)(\rho + \Delta\rho) \Delta x \Delta y \Delta t \end{aligned} \quad (2.39)$$

Eliminating ΔM from **equations 2.38 and 2.39**, dividing through by $\Delta x \Delta y \Delta z \Delta t$ and rearranging gives

$$\begin{aligned} \frac{(\rho + \Delta\rho)(n + \Delta n) - \rho n}{\Delta t} &= - \frac{(q_x + \Delta q_x)(\rho + \Delta\rho) - q_x \rho}{\Delta x} \\ &= - \frac{(q_y + \Delta q_y)(\rho + \Delta\rho) - q_y \rho}{\Delta y} \\ &= - \frac{(q_z + \Delta q_z)(\rho + \Delta\rho) - q_z \rho}{\Delta z} \end{aligned} \quad (2.40)$$

Now letting $\Delta x \Delta y \Delta z$ and Δt tend to zero gives:

$$\frac{\partial(\rho n)}{\partial t} = - \frac{\partial(\rho q_x)}{\partial x} - \frac{\partial(\rho q_y)}{\partial y} - \frac{\partial(\rho q_z)}{\partial z} \quad (2.41)$$

This can be written,

$$\frac{\partial(\rho n)}{\partial t} = -\nabla \cdot (\rho \mathbf{q}) \quad (2.42)$$

2.6 The Storage of Fluid within the Aquifer

Aquifers act both as water conduits and as reservoirs. Under steady state flow conditions, the quantity of water stored in the aquifer remains constant in time. Water flowing from a volume of aquifer is replaced by an equal quantity of incoming water. However, recharge to an aquifer and abstractions from it are generally time-varying. In such cases the amount of water stored in the aquifer changes over time.

When water is pumped from a phreatic aquifer, water is released from storage by the lowering of the water table, and the volume of water released is equal to the volume of extra void spaces produced by this lowering.

When a confined aquifer is pumped, water is released due to the effects of reducing the pressure within the aquifer. This reduction in pressure may result in the compaction of the aquifer, reducing the void space and squeezing out the groundwater like water from a sponge. If the pressures in the aquifer are high enough to compress the groundwater itself then reducing the pressure will result in the groundwater increasing in volume, causing an increased yield. If the piezometric surface is drawn down below the top of the aquifer then the aquifer will behave locally like a phreatic one. In reality, dewatering of an aquifer often occurs as a result of combination of these factors.

It is important in resource planning to understand how much water that is produced by initial pumping comes from a transient dewatering of the aquifer and how much is sustainable for longer periods.

We concentrate here on confined aquifers. The dewatering of phreatic aquifers due to the lowering of the water table will be discussed later.

The term on the left hand side of the continuity equation (**equation 2.42**) expresses how the storage of fluid in the control volume within the saturated part of the aquifer changes with time. This term can be expanded to give:

$$\frac{\partial(\rho n)}{\partial t} = \rho \frac{\partial n}{\partial t} + n \frac{\partial \rho}{\partial t} \quad (2.43)$$

The derivation in first term on the right hand side of **equation 2.43** expresses the rate of change of porosity with respect to time and this change in porosity is generally taken to be caused by the compaction of the aquifer as water is withdrawn and the pressure reduced. The remaining derivative expresses the rate of change of fluid density with respect to time which may also be a function of the releases of pressure and the compressibility of the fluid. It may also be a function of density changes due to temperature or solute concentration.

2.6.1 Storage Changes Due to Aquifer Compaction

The external load on an aquifer is absorbed partly by the solid material and partly by the fluid pressure. Thus, the total stress on the aquifer σ_T is the sum of the effective stress σ_e and the groundwater pressure:

$$\sigma_T = \sigma_e + P \quad (2.44)$$

Since the external load, and hence the total stress, are assumed to be constant we have that

$$\frac{\partial \sigma_e}{\partial t} = -\frac{\partial P}{\partial t} \quad (2.45)$$

It is generally assumed that the relative decreases in total volume are proportional to the increase in effective stress (see **equation 2.13**).

$$-\frac{\Delta V_T}{V_T} = \alpha \Delta \sigma_e \quad (2.46)$$

where α is the coefficient of compressibility of that part of the aquifer

Dividing by Δt and letting Δt tend to zero gives

$$\frac{1}{V_T} \frac{\partial V_T}{\partial t} = -\alpha \frac{\partial \sigma_e}{\partial t} \quad (2.47)$$

The total volume, V_T , of a portion of aquifer is the sum of the volume of the solids, V_s , and the volume of groundwater (or void space), V_w . The bulk porosity can be expressed in terms of these volumes, and since we assume here that the solids comprising the aquifer are effectively incompressible, we have from definition of porosity that

$$\begin{aligned} \frac{\partial n}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{V_T - V_s}{V_T} \right) \\ &= -V_s \frac{\partial}{\partial t} \left(\frac{1}{V_T} \right) \\ &= \frac{V_s}{V_T^2} \frac{\partial V_T}{\partial t} \end{aligned} \quad (2.48)$$

Now combining **equations 2.47** and **2.48** gives

$$\begin{aligned}\frac{\partial n}{\partial t} &= \frac{V_s}{V_T} \alpha \frac{\partial P}{\partial t} \\ &= (1-n) \alpha \frac{\partial P}{\partial t}\end{aligned}\tag{2.49}$$

Thus the expression

$$\rho(1-n) \alpha \frac{\partial P}{\partial t}\tag{2.50}$$

Represents the rate of change due to aquifer compaction.

Typical values of α are given by Freeze and Cherry for a variety of rocks are reproduced in **Table 2.1**.

De Marsily (1986) gives an account incorporating the effect of compaction of the solid material. However the coefficient of compressibility of the solid material within an aquifer is generally very low (de Marsily gives a value of quartz as $2 \times 10^{-11} \text{ m}^2\text{N}^{-1}$) and is thus the effect of the compaction of solids is generally ignored.

It should be noted here that the changes in storage due to aquifer compaction is not a truly reversible process and that although pumping water into an aquifer will increase the fluid pressure and so decreases the effective stress, the aquifer will not reflate. In other words the aquifer is not truly elastic and the Hooke's Law type **equation 2.46** is really only valid when the effective stress is increasing. However, it is usual to use the same storage coefficient when pumping from or into an aquifer.

2.6.2 Storage Changes Due to Compressibility of the Fluid

For constant temperature systems the second term on the right hand side of **equation 2.43** can be further expanded to give

$$n \frac{\partial \rho}{\partial t} = n \rho \beta \frac{\partial P}{\partial t} + n \rho \beta_c \frac{\partial c}{\partial t}$$

Note that,

$$\begin{aligned}\beta &= \frac{\partial \rho / \rho}{\partial P} \Rightarrow \partial \rho = \rho \times \beta \times \partial P \\ \Rightarrow \frac{\partial \rho}{\partial t} &= \rho \beta \left(\frac{\partial P}{\partial t} \right)\end{aligned}\tag{2.51}$$

where

β is the coefficient of compressibility of the fluid, and

β_c is the coefficient of density change as a function of contaminant concentration change.

Thus,

$$\begin{aligned}\frac{\partial}{\partial t}(\rho n) &= n \frac{\partial \rho}{\partial t} + \rho \frac{\partial n}{\partial t} \\ \frac{\partial(\rho n)}{\partial t} &= \rho [n\beta + (1-n)\alpha] \frac{\partial P}{\partial t} + n \rho \beta_c \frac{\partial c}{\partial t}\end{aligned}\tag{2.52}$$

The value of β for water is about $5 \times 10^{-10} \text{ m}^2\text{N}^{-1}$. The value of β_c is a function of the contaminant. For practical purposes we assume that the last term in **equation 2.52** is zero and we define the specific storativity S_s [L^{-1}] by

$$S_s = \rho g [n\beta + (1-n)\alpha] \quad (2.53)$$

from which

$$\frac{\partial(\rho n)}{\partial t} = \frac{S_s}{g} \frac{\partial P}{\partial t} \quad (2.54)$$

The specific storativity represents the volume of water released per unit volume of aquifer per unit decreases in pressure.

2.6.3 The Final Form of the Continuity Equation

We are in a position to formulate the continuity equation in terms of fluid pressure. Using **equations 2.42** and **2.54** gives

$$S_s \frac{\partial P}{\partial t} = -g \nabla \cdot (\rho q) \quad (2.55)$$

If the density of the fluid is homogeneous and constant the continuity equation can be written in terms of piezometric head instead of pressure. From the definition of piezometric head **equation 2.2** we have that

$$\rho g \frac{\partial h}{\partial t} = \frac{\partial P}{\partial t} \quad (2.56)$$

since ρ is constant.

So **equation 2.55** becomes

$$S_s \rho \frac{\partial h}{\partial t} = -\nabla \cdot (\rho q) \quad (2.57)$$

Since the fluid is homogeneous the continuity equation becomes

$$S_s \frac{\partial h}{\partial t} = -\nabla \cdot q \quad (2.58)$$

2.7 The Three-Dimensional Equations of Groundwater Flow

2.7.1 Flow in Confined Aquifers

Having established the mathematical formulations of Darcy's Law and the mass balance or continuity equation, it is a simple matter to develop the equation describing three-dimensional fluid flow through a porous medium. In the case of density dependent flow we must use a formulation based upon pressure. Substituting the expression for the specified discharge q given by Darcy's Law (**equation 2.30**) into the pressure formulation of the continuity equation (**equation 2.55**) gives

$$S_s \frac{\partial P}{\partial t} = \nabla \cdot \left[\frac{\rho g k}{\mu} (\nabla P + \rho g \nabla z) \right] \quad (2.59)$$

In the simpler case of a homogeneous fluid with constant density we can use a formulation in terms of head. Substituting for \mathbf{q} from **equation 2.31** into **equation 2.58** gives

$$S_s \frac{\partial h}{\partial t} = \nabla \cdot (K \nabla h) \quad (2.60)$$

which can be written

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial h}{\partial z} \right) \quad (2.61)$$

If the aquifer is isotropic this becomes

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K \frac{\partial h}{\partial z} \right) \quad (2.62)$$

If it is also homogeneous the equation becomes

$$S_s \frac{\partial h}{\partial t} = K \left[\left(\frac{\partial^2 h}{\partial x^2} \right) + \left(\frac{\partial^2 h}{\partial y^2} \right) + \left(\frac{\partial^2 h}{\partial z^2} \right) \right] \quad (2.63)$$

2.7.2 Flow in Layered Media

In this section we investigate groundwater flow through a class of aquifer formations with anisotropic permeability. These are aquifers composed of a number of layers of rocks with different permeabilities. Unless otherwise stated, we assume for simplicity that the layers of rock are horizontal. However the major results of this section apply equally to strata which are not horizontal.

The Effective Hydraulic conductivity Parallel to the Layers

Consider the horizontal flow through a stratified with N layers with hydraulic conductivity K_1, K_2, \dots, K_N and thickness b_1, b_2, \dots, b_N . Then for each layer, Darcy's Law gives

$$Q_i = -b_i K_i \frac{(h_1 - h_2)}{L} \quad (2.64)$$

Since the discharge through the aquifer must equal the sum of the discharges through each layer we have that the total discharge is given by

$$Q = \sum_{i=1}^N Q_i \quad (2.65)$$

Using the total discharge and an effective (average) horizontal hydraulic conductivity K_h^* , Darcy's Law gives

$$Q = -b K_h^* \frac{(h_1 - h_2)}{L} \quad (2.66)$$

Substituting for Q_i in **equation 2.65** and eliminating Q from **equations 2.65** and **2.66** gives

$$-b K_h^* \frac{(h_1 - h_2)}{L} = -\sum_{i=1}^N b_i K_i \frac{(h_1 - h_2)}{L} \quad (2.67)$$

Whence

$$K_h^* = \frac{\sum_{i=1}^N b_i K_i}{b} \quad (2.68)$$

Which is the weighted arithmetic mean of the hydraulic conductivities. In terms of transmissivity this is

$$T^* = \sum_{i=1}^N T_i \quad (2.69)$$

The Effective Hydraulic conductivity at Right Angles to the Layers

Consider now the vertical flow across the layers of the aquifer. Let Δh_i be the head drop across the i 'th layer. Since mass continuity demands that the discharge across each layer be the same, using Darcy's Law, we get that

$$Q = K_i \frac{\Delta h_i}{b_i} \quad \text{for } i = 1, \dots, N \quad (2.70)$$

where

$$\sum_{i=1}^N \Delta h_i = h_1 - h_2 \quad (2.71)$$

Using the effective hydraulic vertical conductivity K_v^* in Darcy's Law gives

$$Q = K_v^* \frac{(h_1 - h_2)}{b} \quad (2.72)$$

From **equations 2.70** and **2.71**

$$h_1 - h_2 = -\sum_{i=1}^N \frac{Q b_i}{K_i} \quad (2.73)$$

Substituting for $h_1 - h_2$ in **equation 2.72** and rearranging gives

$$K_v^* = \frac{b}{\sum_{i=1}^N \frac{b_i}{K_i}} \quad (2.74)$$

This is the weighted harmonic mean of the hydraulic conductivities.

2.8 The Two-Dimensional Equations of Groundwater Flow

The lateral extent of a regional aquifer is usually much greater than its thickness. This disparity between the vertical and horizontal dimensions of many aquifers has resulted in the adoption of two-dimensional equations for the description of regional groundwater flow. In this case, the vertical flow components within the body of the aquifer are effectively ignored, and flow is assumed to be horizontal everywhere. In phreatic aquifers, this is known as the **Dupuit Hypothesis**. This is acceptable as long as the pressure changes due to vertical flow are small compared with those arising from horizontal flow. This is generally true in aquifers at some distance (greater than twice the thickness of the aquifer) from point sources or sinks. In the neighborhood of wells or partially penetrating streams, vertical flow components may be significant. However it is possible to introduce corrections in these zones if necessary in the development of the regional flow equation through the adoption of suitable approximations. The cases of confined and phreatic aquifers have to be considered separately. In the following development we assume that the fluid is homogeneous with constant density.

2.8.1 The Horizontal Flow Assumptions and Darcy's Law

The assumption of horizontal flow allows the development of a vertically integrated form of Darcy's law. In other words we can use a version of Darcy's law which represents the fluid flow over the full depth of an aquifer rather than just at a point in three-dimensional space.

For simplicity we consider flow in the x direction only. In this case Darcy's law is

$$q_x = -K \frac{\partial h}{\partial x} \quad (2.75)$$

We now integrate over the vertical dimension to give, Q_x , the discharge per unit width of aquifer.

$$Q_x = - \int_{z_1}^{z_2} K_x \frac{\partial h}{\partial x} dz \quad (2.76)$$

Since we assume that flow is horizontal it follows that we assume also that h is constant over depth and hence that $\frac{\partial h}{\partial x}$ is constant over depth. Hence we can write

$$Q_x = - \left(\int_{z_1}^{z_2} K_x dz \right) \frac{\partial h}{\partial x} \quad (2.77)$$

2.8.2 Horizontal Flow in Confined Aquifers

In confined aquifers the lower and upper limits of the integral refer to the elevation of the base and the top of the aquifer respectively. In this case the integral of hydraulic conductivity over depth is called the transmissivity T of the aquifer at a point. So parallel to the x axis we write Darcy's law as

$$Q_x = -T \frac{\partial h}{\partial x} \quad (2.80)$$

and in both horizontal dimensions as

$$Q = -T_x \frac{\partial h}{\partial x} - T_y \frac{\partial h}{\partial y} \quad (2.81)$$

If in addition we assume that the hydraulic conductivity is constant over depth then the transmissivity is just the product of the hydraulic conductivity and the depth of the aquifer at a point, i.e.

$$T = Kb \quad (2.82)$$

We also define a new storage coefficient, S , called storativity given by

$$S = bS_s \quad (2.83)$$

which represents the volume of water released from a unit area of aquifer per unit decrease in head.

Substituting for Q_x and Q_y in the continuity **equations 2.61 & 2.62** using Darcy's law (assuming that the principal axes of permeability are aligned with the coordinates axes) gives

$$S \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(T_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(T_y \frac{\partial h}{\partial y} \right) \quad (2.84)$$

If the transmissivity is isotropic the equation becomes

$$S \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(T \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(T \frac{\partial h}{\partial y} \right) \quad (2.85)$$

If in addition the transmissivity is homogeneous the equation becomes

$$S \frac{\partial h}{\partial t} = T \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) \quad (2.86)$$

Finally, if the groundwater flow system is in steady state, that is the head values are not varying over time, then the equation reduces to the Laplace Equation

$$\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = 0 \quad (2.87)$$

2.8.3 Horizontal Flow in Unconfined Aquifers

The approach to the development of the two-dimensional flow equation for unconfined aquifers is the same as in the confined case, based upon the use of a control volume. A conceptual model of a leaky confined aquifer system is shown in **Figure 2.10**. The essential difference between the cases is that, whereas in the confined case the control volume is defined by the geometry of the aquifer, in the unconfined case, the top of the control volume coincides with the water table and, therefore, rises and falls with it.

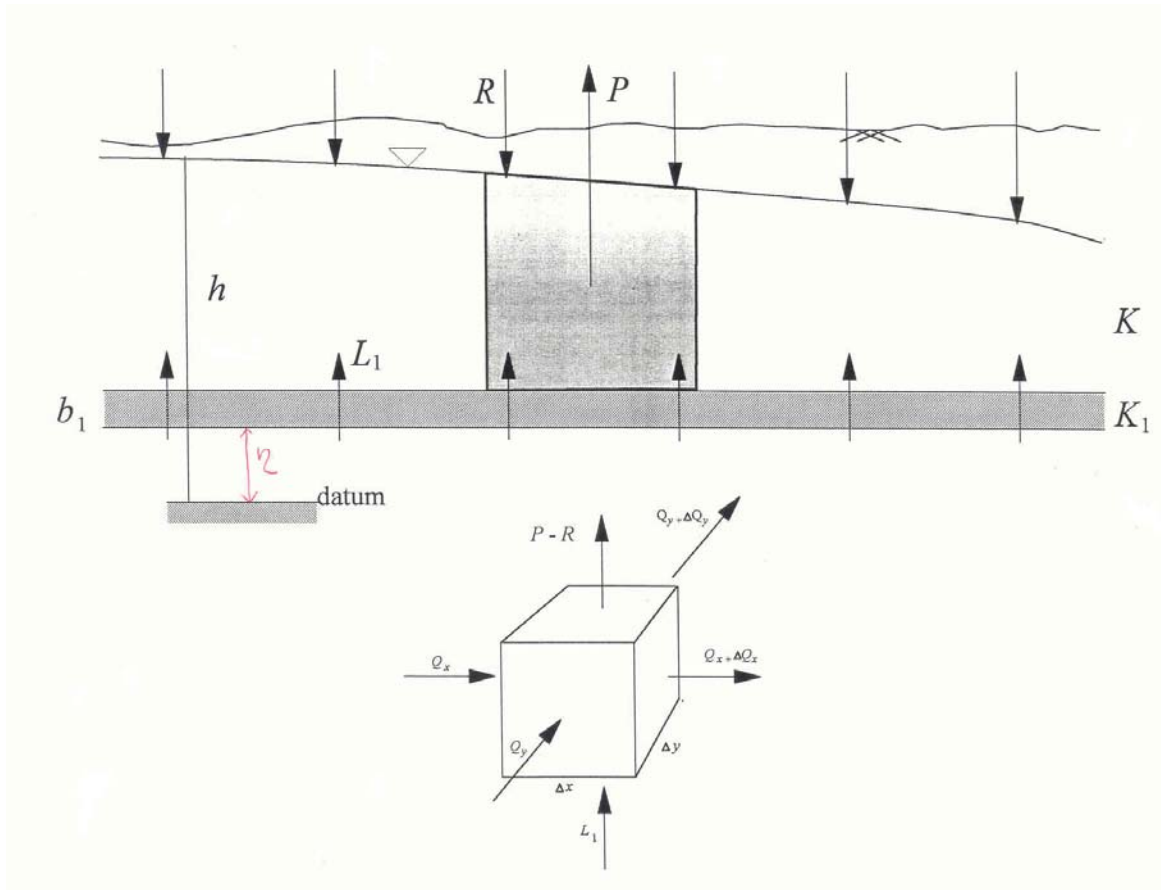


Figure 2.10

The limits of the integral in the integrated from Darcy's law now correspond to the elevation of the bottom of the aquifer and to the elevation of the water table, and so we get

$$Q_x = - \left(\int_{\eta}^h K_x dz \right) \frac{\partial h}{\partial x} \quad (2.88)$$

where η is the elevation of the base of the aquifer

Since h is not constant, we cannot define a constant transmissivity for unconfined aquifers. However, if K is constant over depth we can write

$$Q_x = - K_x (h - \eta) \frac{\partial h}{\partial x} \quad (2.89)$$

The specific yield S_y is defined as the volume of water released from a unit area of an unconfined aquifer due to a unit decrease in the water table. To emphasize that the specific yield is not the same as the porosity, some authors introduced the specific retention S_R which is defined as the difference between total porosity and the specific yield.

Substituting for Q_x and Q_y in the continuity equation using Darcy's law (assuming that the principal axes of permeability are aligned with the coordinates axes) gives

$$S_Y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K_x (h - \eta) \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y (h - \eta) \frac{\partial h}{\partial y} \right) \quad (2.90)$$

If the permeability is isotropic the equation becomes

$$S_Y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(K (h - \eta) \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K (h - \eta) \frac{\partial h}{\partial y} \right) \quad (2.91)$$

For homogeneous conditions the equation becomes

$$S_Y \frac{\partial h}{\partial t} = K \left[\frac{\partial}{\partial x} \left((h - \eta) \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left((h - \eta) \frac{\partial h}{\partial y} \right) \right] \quad (2.92)$$

These are all forms of the Boussinesq Equation.

It is important to note that unlike the equations describing horizontal groundwater flow in confined aquifers, those describing horizontal flow in unconfined aquifers are not linear since they contain terms involving the product of the state variable h and its spatial derivations. This means that they must be solved iteratively which can be very time consuming. Some computer codes employ a technique of linearization of the equations to cut down on computer time. This involves replacing $(h - \eta)$ by an average saturated thickness b . This is only justifiable if the lateral extent of the aquifer is much greater than the saturated thickness.

2.9 Further Treatments of Groundwater Flow Equations

2.9.1 Confined Aquifer

If there is the steady movement of groundwater in a confined aquifer, there will be a gradient or slope to the potentiometric surface of the aquifer. Likewise, we know that the water will be moving in the opposite direction of grad h . For flow of this type, Darcy's law may be used directly. In **Figure 2.8**, a portion of a confined aquifer of uniform thickness is shown. The potentiometric surface has a linear gradient; i.e., its two-directional projection is a straight line. There are two observation wells where the hydraulic head can be measured.

The quantity of flow per unit width, q , may be determined from Darcy's law:

$$q = -Kb \frac{dh}{dl} \quad (2.93)$$

where,

q	is the flow per unit width (m ² /day)
K	is the hydraulic conductivity (m/day)
b	is the aquifer thickness (m)
$\frac{dh}{dl}$	is the slope of potentiometric surface (dimensionless)

One may wish to know the head, h (m), at some intermediate distance, x (m), between h_1 and h_2 . This may be found from the equation.

$$h = h_1 - \frac{q}{Kb} x \quad (2.94)$$

where, x is the distance from h_1 .

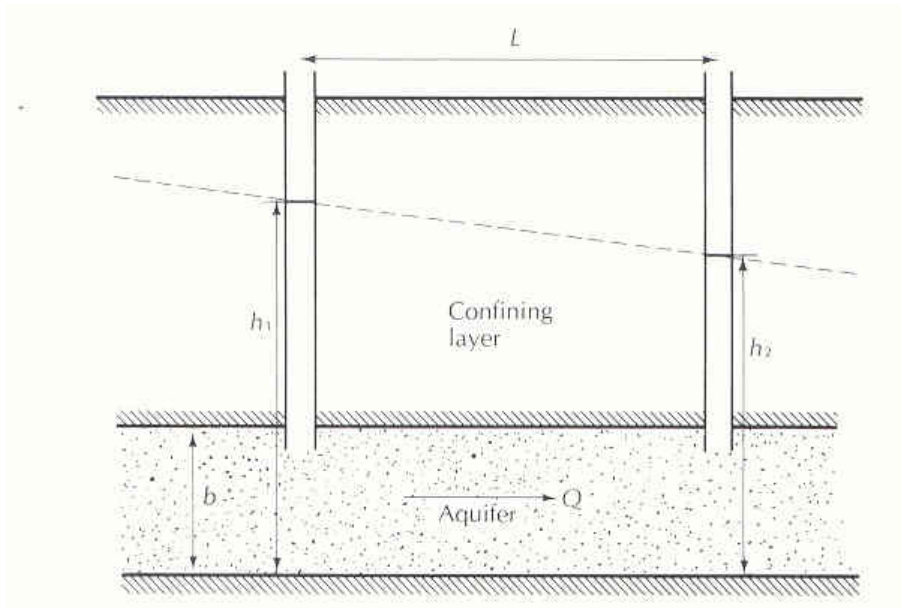


Figure 2.8 Steady flow through a confined aquifer of uniform thickness

2.9.2 Unconfined Aquifer

In an unconfined aquifer, the fact that the water table is also the upper boundary of the region of flow complicates flow determinations. **Figure 2.9** illustrates the problem. On the left side of the figure, the saturated flow region is h_1 feet thick. On the right side, it is h_2 feet thick, which is $h_1 - h_2$ feet thinner than the left side. If there is no recharge or evaporation as the flow traverses the region, the quantity of water flowing through the left side is equal to that flowing through the right side. From Darcy's law, it is obvious that since the cross-sectional area is smaller on the right side, the hydraulic gradient must be greater. Thus, the gradient of the water table in unconfined flow is not constant; it increases in the direction of flow.

This problem was solved by Dupuit, and his assumptions are known as the **Dupuit assumptions**. The assumptions are that (1) the hydraulic gradient is equal to the slope of the water table and (2) for small water-table gradients, the streamlines are horizontal and the equipotential lines are vertical. Solutions based on these assumptions have proved to be very useful in many practical problems. However, the Dupuit assumptions do not allow for a seepage face above the outflow side.

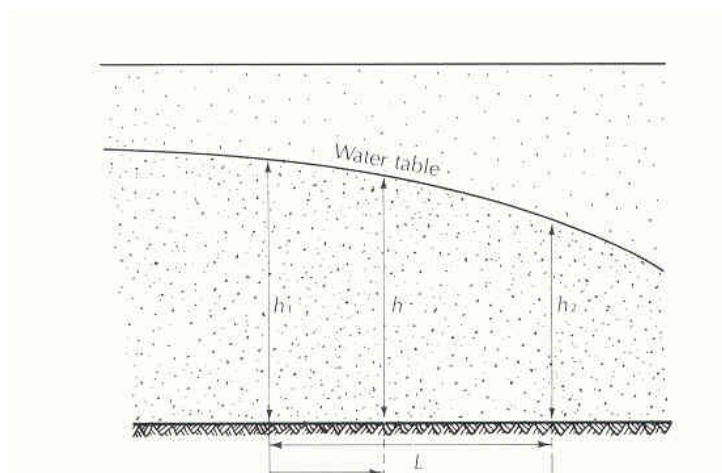


Figure 2.9 Steady flow through an unconfined aquifer resting on a horizontal impervious surface

From Darcy's law,

$$q = -Kh \frac{dh}{dx} \quad (2.95)$$

where h is the saturated thickness of the aquifer. At $x=0$, $h = h_1$; at $x=L$, $h = h_2$

Equation 2.95 may be set up for the integration with the boundary conditions:

$$\int_0^L q dx = -K \int_{h_1}^{h_2} h dh \quad (2.96)$$

Integration of the preceding yields:

$$qL = -K \left(\frac{h_2^2}{2} - \frac{h_1^2}{2} \right) \quad (2.97)$$

Rearrangement of **equation 2.97** yields the Dupuit equation

$$q = \frac{1}{2} K \left(\frac{h_1^2 - h_2^2}{L} \right) \quad (2.98)$$

If we consider a small prism of the unconfined aquifer, it will have the shape of **Figure 2.10**. On one side it is h units high and slopes in the x -direction. Given the Dupuit assumptions, there is no flow in the z -direction. The flow in the x -direction, per unit width, is q_x . From Darcy's law, the total flow in the x -direction through the left face of the prism is

$$q_x dy = -K \left(h \frac{\partial h}{\partial x} \right)_x dy \quad (2.99)$$

where dy is the width of the face of the prism.

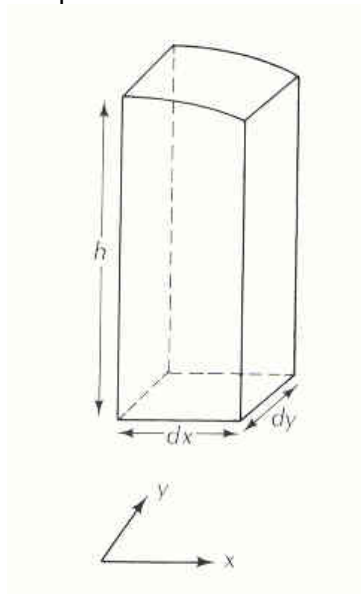


Figure 2.10 Control volume for flow through a prism of an unconfined aquifer with the bottom resting on a horizontal impervious surface and the top coinciding with the water table.

The discharge through the right face, q_{x+dx} is

$$q_{x+dx} dy = -K \left(h \frac{\partial h}{\partial x} \right)_{x+dx} dy \quad (2.100)$$

Note that $\left(h \frac{\partial h}{\partial x} \right)$ has different values at each face. The change in flow rate in the x-direction between the two faces is given by

$$(q_{x+dx} - q_x) dy = -K \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) dx dy \quad (2.101)$$

Through a similar process, it can be shown that the change in the flow rate in the y-direction is

$$(q_{y+dy} - q_y) dx = -K \frac{\partial}{\partial y} \left(h \frac{\partial h}{\partial y} \right) dy dx \quad (2.102)$$

For steady flow, any change in flow through the prism must be equal to a gain or loss of water across the water table. This could be infiltration or evapotranspiration. The net addition or loss is at a rate of w , and the volume change within the initial volume is $w dx dy$ where $dx dy$ is the area of the surface. If w represents evapotranspiration, it will have a negative value. As the change in flow is equal to the new addition,

$$-K \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) dx dy - K \frac{\partial}{\partial y} \left(h \frac{\partial h}{\partial y} \right) dy dx = w dx dy \quad (2.103)$$

We can simplify **equation 2.103** by dropping out $dx dy$ and combining the differentials:

$$-K \left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right) = 2w \quad (2.104)$$

If $w = 0$, then **equation 2.104** reduces to a form of the Laplace equation:

$$\left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} \right) = 0 \quad (2.105)$$

If flow is in only one direction and we align the x-axis parallel to the flow, then there is no flow in the y-direction, and equation **2.104 becomes**

$$\frac{d^2(h^2)}{dx^2} = -\frac{2w}{K} \quad (2.106)$$

Integration of this equation yields the expression

$$h^2 = -\frac{wx^2}{K} + c_1 x + c_2 \quad (2.107)$$

where c_1 and c_2 are constants of integration.

The following boundary conditions can be applied: at $x=0$, $h = h_1$; at $x=L$, $h = h_2$ (**Figure 2.11**). By substituting these into **equation 2.107**, the constants of integration can be evaluated with the following result:

$$h^2 = h_1^2 - \frac{(h_1^2 - h_2^2)x}{L} + \frac{w}{K}(L-x)x \quad (2.108)$$

or

$$h = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)x}{L} + \frac{w}{K}(L-x)x} \quad (2.109)$$

where,

- h is head at x (m)
- K is the hydraulic conductivity (m/day)
- x is the distance from the origin(m)
- h_1 is the head at the origin(m)
- h_2 is the head at L (m)
- L is the distance from the origin at the point h_2 is measured (m)
- w is the recharge rate (m/day)

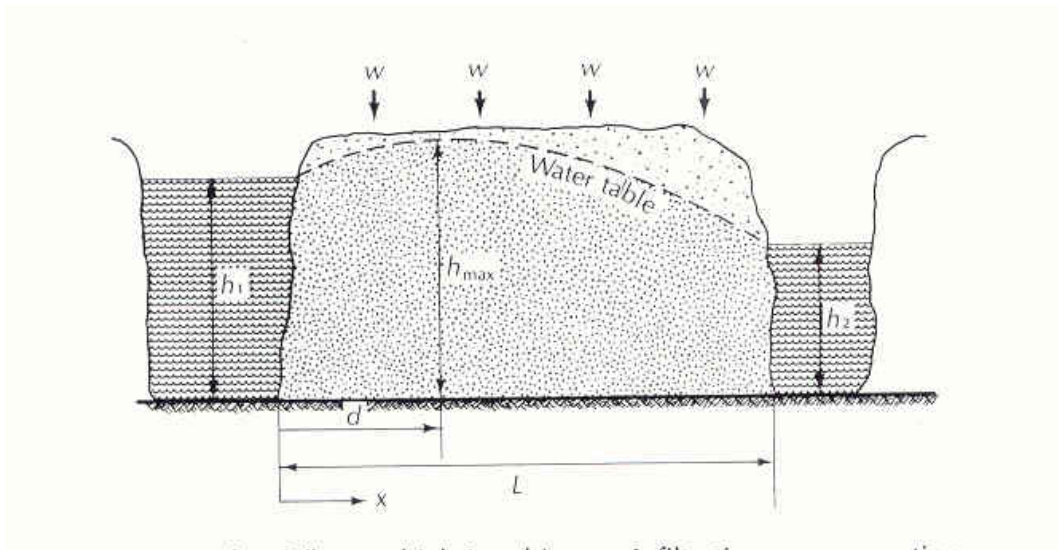


Figure 2.11 Unconfined flow, which is subject to infiltration or evaporation

This equation can be used to find the elevation of the water table anywhere between two points located L distance apart if the saturated thickness of the aquifer is known at the two end points.

For the case in which there is no infiltration or evaporation, $w=0$ and **equation 2.109** reduces to

$$h = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)x}{L}} \quad (2.110)$$

By differentiating **equation 2.108**, and because $q_x = -K h \frac{dh}{dx}$, it may be shown that the discharge per unit width, q_x , at any section x distance from the origin is given by:

$$q_x = \frac{K(h_1^2 - h_2^2)}{2L} - w \left(\frac{L}{2} - x \right) \quad (2.111)$$

If the water table is subject to infiltration, there may be a water divide with a crest in the water table. In this case, q_x will be zero at the water divide. If d is the distance from the origin to a water divide, then substituting $q_x = 0$ and $x = d$ into **equation 2.110** yields:

$$d = \frac{L}{2} - \frac{K}{w} \frac{(h_1^2 - h_2^2)}{2L} \quad (2.112)$$

Once the distance from the origin to the water divide has been found, then the elevation of the water divide may be determined by substituting d for x in **equation 2.109**

$$h_{\max} = \sqrt{h_1^2 - \frac{(h_1^2 - h_2^2)d}{L} + \frac{w}{K}(L-d)d} \quad (2.113)$$