## Example 2.1: Piezometric Head

Figure 2.17 shows a dam, using equation 2.2: $h=\frac{P}{\rho g}+z$, show that the Piezometric head on surface $A B=H_{1}$ at any point on surface $A B$, (i.e. show that $h_{1}=h_{2}=h_{3}=H_{1}$ ).


Figure 2.17

## Answer 2.1

From equation 2.2: $h=\frac{P}{\rho g}+z$

$$
\begin{array}{ll}
h_{1}=\frac{P_{1}}{\rho g}+z_{1} \quad, \quad h_{2}=\frac{P_{2}}{\rho g}+z_{2}, & h_{3}=\frac{P_{3}}{\rho g}+z_{3} \\
P_{1}=\rho g d_{1}, z_{1}=H_{1}-d_{1} \quad, \quad P_{2}=\rho g d_{2}, z_{2}=H_{1}-d_{2} \quad, \quad P_{3}=\rho g d_{3}, z_{3}=H_{1}-d_{3}
\end{array}
$$

Then,

$$
\begin{aligned}
& h_{1}=\frac{P_{1}}{\rho g}+z_{1}=\frac{\rho g d_{1}}{\rho g}+H_{1}-d_{1}=d_{1}+H_{1}-d_{1}=H_{1} \\
& h_{2}=\frac{P_{2}}{\rho g}+z_{2}=\frac{\rho g d_{2}}{\rho g}+H_{1}-d_{2}=d_{2}+H_{1}-d_{2}=H_{1} \\
& h_{3}=\frac{P_{3}}{\rho g}+z_{3}=\frac{\rho g d_{3}}{\rho g}+H_{1}-d_{3}=d_{3}+H_{1}-d_{3}=H_{1}
\end{aligned}
$$

So, on the surface $A B h=H_{1}$, but the pressure is not the same.

## Example 2.2: Darcy's Law and Groundwater velocity

A confined aquifer is 50 m thick and 0.5 km wide. Two observation wells are located 1.4 km apart in the direction of flow. Head in well No. 1 is 50 m and in well No. 2 is 42 m . Hydraulic conductivity is $0.7 \mathrm{~m} /$ day. Effective porosity is 0.2 .
(i) What is the total flow of water through the aquifer?
(ii) What is the actual groundwater velocity between the two wells?

## Answer 2.2

(i) From equation 2.4, $Q=-K A \frac{h_{2}-h_{1}}{L}$

Where, $K=0.7 \mathrm{~m} /$ day, $A=50 \times 500=25,000 \mathrm{~m}^{2}, h_{1}=50 \mathrm{~m}, h_{2}=42 \mathrm{~m}, L=1,400 \mathrm{~m}$.
So, the total flow of water through the aquifer $=Q=-0.7 \times 25,000 \times \frac{-8}{1,400}=100 \mathrm{~m}^{3} / \mathrm{day}$.
(ii) From equation 2.6, $u=\frac{K}{n_{e}} \times \frac{\Delta h}{\Delta l}$

So, the actual groundwater velocity between the two wells $=u=\frac{0.7}{0.2} \times \frac{8}{1400}=0.02 \mathrm{~m} / \mathrm{day}$.

## Example 2.3: Specific Discharge

Figure 2.18 shows two piezometers in a confined aquifer:
(i) What is the direction of the groundwater flow in the Figure?
(ii) What is the specific discharge using Darcy's law, when the hydraulic conductivity is $2 \times 10^{-3} \mathrm{~m} / \mathrm{sec}$ ?


Figure 2.18

## Answer 2.3

(i) The direction of the flow is
(ii) From equation 2.5, $q=-K \frac{d h}{d x}$.

So the specific discharge $=q=-2 \times 10^{-3} \frac{-12}{500}=4.8 \times 10^{-5} \mathrm{~m} / \mathrm{sec}$.

## Example 2.4: Groundwater Flow into a river

A river penetrates a confined aquifer of 10 m thick Figure 2.19. A long drought decreases the flow in the stream by $0.5 \mathrm{~m}^{3} / \mathrm{sec}$ between gauging stations. $A$ and $B$ some 6000 m apart. On the west side of the river, the piezometer contours parallel the bank and slope towards the river at $0.0004 \mathrm{~m} / \mathrm{m}$. The piezometric contours on the east side of the river slope away from the river toward a well field at a slope of $0.0006 \mathrm{~m} / \mathrm{m}$.


Figure 2.19
(i) Explain the flow system along this section.
(ii) Using Darcy's law and the continuity equation, compute the transmissivity of the aquifer through the river section.

## Answer 2.4

(i) Flow into the river from the aquifer occurs on the west and flow from the river into the aquifer occurs on the east.
(ii) From continuity equation, we have:

$$
Q_{i n}-Q_{o u t}=\Delta Q
$$

Due to the slope of the piezometric surface, we assume the flow into the stream from the aquifer occurs on the west and flow from the stream into the aquifer occurs at the east, or:

$$
Q_{\text {in }}=K A(d h / d l)_{\text {west }} \quad \text { and } \quad Q_{\text {out }}=K A(d h / d l)_{\text {east }}
$$

Thus,

$$
\begin{aligned}
& \left.K A(d h / d l)_{\text {west }}-K A(d h / d l)\right)_{\text {east }}=-0.5 \mathrm{~m}^{3} / \mathrm{sec} \quad(\text { Decrease }) \\
\Rightarrow & K A\left((d h / d l)_{\text {west }}-(d h / d l)_{\text {east }}\right)=-0.5 \mathrm{~m}^{3} / \mathrm{sec} \\
\Rightarrow & K(6000 \mathrm{~m} \times 10 \mathrm{~m})[0.0004 \mathrm{~m} / \mathrm{m}-0.0006 \mathrm{~m} / \mathrm{m}]=-0.5 \mathrm{~m}^{3} / \mathrm{sec} \\
\Rightarrow & K=4.17 \times 10^{-2} \mathrm{~m} / \mathrm{sec}
\end{aligned}
$$

$$
\text { but } T=K b=4.17 \times 10^{-2} \mathrm{~m} / \mathrm{sec} \times 10 \mathrm{~m}=4.17 \times 10^{-1} \mathrm{~m}^{2} / \mathrm{sec}
$$

## Example 2.5: Vertical Hydraulic Conductivity

Three geological formations overlie one another with the characteristics shown in Figure 2.20 below. A constant-velocity vertical flow field exists across the three formations. The hydraulic head is 11 m at the top of the formations and 6.5 m at the bottom.


Figure 2.20
(i) Calculate the average vertical hydraulic conductivity?
(ii) Calculate the velocity through the entire system?
(iii) Calculate the hydraulic head at the two internal boundaries?

## Answer 2.5

(i) According to equation 2.74, $K_{v}=\frac{b}{\sum_{i=1}^{n} b_{i} / K_{i}}$

So, $K_{z}=\frac{280}{\frac{50}{18}+\frac{20}{0.5}+\frac{210}{85}}=6.2 \mathrm{~m} / \mathrm{day}$
(ii) Now, we find the velocity through the entire system.

$$
q_{z}=-K \cdot \frac{\Delta h}{\Delta l}=-6.2 \times \frac{6.5-11}{280}=9.96 \times 10^{-2} \mathrm{~m} / \text { day }
$$

(iii) Now, we use the calculated velocity in (ii) to find the change in head across each layer.

$$
\begin{aligned}
& q_{z}=-K \cdot \frac{\Delta h}{\Delta l}=-K \frac{\left(h_{2}-h_{1}\right)}{b} \Rightarrow q_{z} b=-K \times\left(h_{2}-h_{1}\right) \\
& \Rightarrow h_{2}=-\frac{q_{z} b}{K}+h_{1} \text { or } h_{2}=h_{1}-\frac{q_{z} b}{K}
\end{aligned}
$$

Thus, at the bottom of the first aquifer we get:

$$
\begin{aligned}
& h_{2}=h_{1}-\frac{q_{z} b}{K}=11-\frac{9.96 \times 10^{-2} \times 50}{6.2} \\
& h_{\text {bottom of the first layer }}=10.2 \mathrm{~m} .
\end{aligned}
$$

At the bottom of the second aquifer we get:

$$
\begin{aligned}
& h_{2}=h_{1}-\frac{q_{z} b}{K}=10.2-\frac{9.96 \times 10^{-2} \times 20}{6.2} \\
& h_{\text {bottom of the sec ond layer }}=9.88 \mathrm{~m} .
\end{aligned}
$$

To check our calculations, we continue:

$$
\begin{aligned}
& h_{2}=h_{1}-\frac{q_{z} b}{K}=9.88-\frac{9.96 \times 10^{-2} \times 210}{6.2} \\
& h_{\text {bottomof the third layer }}=6.50 \mathrm{~m} .
\end{aligned}
$$

So, our calculations are right since the head at the bottom of the formations is 6.5 m .

## Example 2.6: Compressibility and Effective Stress

A confined aquifer with an initial thickness of 45 m consolidates (compacts) 0.20 m when the head is lowered by 25 m . The porosity of the aquifer is $12 \%$ after compaction.
(i) What is the vertical compressibility of the aquifer?
(ii) Calculate the storativity of the aquifer?
(iii) How much water was released from storage for a head drop of 25 m averaged over the aquifer, assume that the aquifer has an area of $10^{6} \mathrm{~m}^{2}$ ?

## Answer 2.6

(i) The given parameter values are $d P=25 \mathrm{~m}, b=45 \mathrm{~m}$, and $d b=0.20 \mathrm{~m}$. A pressure head of 25 m of water can be converted to a fluid pressure by multiplying the pressure head by the density of water times the gravitational constant.

$$
d P=25 \mathrm{~m} \times 1000 \mathrm{~kg} / \mathrm{m}^{3} \times 9.8 \mathrm{~m} / \mathrm{s}^{2}=245,000 \mathrm{~N} / \mathrm{m}^{2}
$$

From equation 2.19,

$$
\begin{gathered}
\alpha=\frac{d b / b}{d P} \\
\alpha=\frac{0.2 \mathrm{~m} / 45 \mathrm{~m}}{245,000 \mathrm{~N} / \mathrm{m}^{2}}=1.8 \times 10^{-8} \mathrm{~m}^{2} / \mathrm{N}
\end{gathered}
$$

(ii) Aquifer Storativity is found from equation 1.26

$$
S=b \rho g[\alpha+n \beta]
$$

The given parameter values are $b=44.8 \mathrm{~m}, n=0.12, \rho_{w}=1000 \mathrm{~kg} / \mathrm{m}^{3}, g=9.8 \mathrm{~m} / \mathrm{s}^{2}$ and $\alpha=1.8 \times 10^{-8} \mathrm{~m}^{2} / N$, and $\beta=4.6 \times 10^{-10} \mathrm{~m}^{2} / \mathrm{N}$.

$$
\begin{aligned}
S & =(44.8 \mathrm{~m})\left(1000 \mathrm{~kg} / \mathrm{m}^{3}\right)\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)\left[1.8 \times 10^{-8} \mathrm{~m}^{2} / \mathrm{N}+0.12 \times 4.6 \times 10^{-10} \mathrm{~m}^{2} / \mathrm{N}\right] \\
& =(44.8 \mathrm{~m})\left(9800 \mathrm{~N} / \mathrm{m}^{3}\right)\left(1.806 \times 10^{-8} \mathrm{~m}^{2} / \mathrm{N}\right) \\
& =7.9 \times 10^{-3}
\end{aligned}
$$

(iii) From equation 1.30, $V_{w}=S . A . \Delta h$

The volume of water released from storage of aquifer $=7.9 \times 10^{-3} \times 10^{6} \mathrm{~m}^{2} \times 25 \mathrm{~m}=$ $197.5 \times 10^{3} \mathrm{~m}^{3}$.

## Example 2.7: Groundwater Flow Equations

Define the model assumptions implicit in the following equations of groundwater motion (i.e. state whether the flow is steady or transient; 1-dimenional or 2-dimensional or 3-dimensional; homogeneous or heterogeneous; isotropic or anisotropic).
(i) $K \frac{d}{d x}\left[h \frac{d h}{d x}\right]=0$
(ii) $\frac{\partial^{2} \phi}{\partial x^{2}}+\frac{\partial^{2} \phi}{\partial y^{2}}=0$
(iii) $\frac{\partial}{\partial x_{i}}\left[T_{i j} \frac{\partial \phi}{\partial x_{j}} a\left(x_{i}\right)\right]=S_{o} \frac{\partial \phi}{\partial t}, \quad i=1,2$

## Answer 2.7

(i) Homogenous, isotropic, 1-d, steady.
(ii) Homogenous, isotropic, can be 3-d, steady.
(iii) Heterogeneous, anisotropic, 2-d, transient

## Example 2.8: Groundwater Flow Equations

Show that for constant density and viscosity the following two equations are related:

$$
\rho S_{o p} \frac{\partial P}{\partial t}-\nabla \cdot\left[\left(\frac{\rho k}{\mu}\right)(\nabla p+\rho g)\right]=Q_{p} \quad \text { and } \quad S_{o} \frac{\partial \phi}{\partial t}-K \nabla^{2} \phi=Q^{*}
$$

Hints:

1) $S_{0}=S_{o p} \rho g$,
2) $K=\frac{k \rho g}{\mu}$,
3) $\nabla z=1$,
4) $\phi=\frac{P}{\rho g}+z$,
5) $\frac{Q_{p}}{\rho}=Q^{*}$

## Answer 2.8

$$
\begin{aligned}
& \rho S_{o p} \frac{\partial P}{\partial t}-\nabla \cdot\left[\left(\frac{\rho k}{\mu}\right)(\nabla p+\rho g)\right]=Q_{p} \quad \Rightarrow \rho S_{o p} \frac{\partial P}{\partial t}-\nabla \cdot\left[\left(\frac{\rho k}{\mu}\right)\left(\rho g \frac{\nabla p}{\rho g}+\rho g\right)\right]=Q_{p} \\
& \Rightarrow \rho S_{o p} \frac{\partial P}{\partial t}-\nabla \cdot\left[\rho g\left(\frac{\rho k}{\mu}\right)\left(\frac{\nabla p}{\rho g}+1\right)\right]=Q_{p} \quad \Rightarrow \rho S_{o p} \frac{\partial P}{\partial t}-\nabla \cdot\left[\rho g\left(\frac{\rho k}{\mu}\right) \nabla\left(\frac{p}{\rho g}+z\right)\right]=Q_{p} \\
& \Rightarrow \rho S_{o p} \frac{\partial P}{\partial t}-\nabla \cdot\left[\rho\left(\frac{\rho g k}{\mu}\right) \nabla \phi\right]=Q_{p},\left(\text { Note that } 1=\nabla z, \phi=\frac{p}{\rho g}+z, \text { and } \frac{\rho g k}{\mu}=K\right) \\
&\text { (divide all terms by cons tant } \left.\rho \Rightarrow S_{o p} \frac{\partial P}{\partial t}-\nabla[K \nabla \phi]=\frac{Q_{p}}{\rho} \quad \text { (Note that } \frac{Q_{p}}{\rho}=Q^{*}\right) \\
& \text { but } P=\rho g(\phi-z) \Rightarrow \frac{\partial P}{\partial t}=\rho g \frac{\partial \phi}{\partial t} \\
& \Rightarrow S_{o p} \rho g \frac{\partial \phi}{\partial t}-K \nabla^{2} \phi=Q^{*},\left(\text { Note that } S_{o}=S_{o p} \rho g\right) \\
& \Rightarrow S_{0} \frac{\partial \phi}{\partial t}-K \nabla^{2} \phi=Q^{*}
\end{aligned}
$$

## Example 2.9: Groundwater Flow Equations

Given the piezometric heads in three observation wells located in a homogeneous confined aquifer of constant transmissivity, $T=5000 \mathrm{~m}^{2} /$ day.

| Well | $X(m)$ | $Y(m)$ | $\Phi(m)$ |
| :---: | :---: | :---: | :---: |
| A | 0 | 0 | 15 |
| B | 0 | 300 | 10.4 |
| $C$ | 200 | 0 | 12.1 |

(i) Draw the contours of the piezometric surface
(ii) Determine the discharge through the aquifer per unit width (magnitude and direction).
(iii) For the same data above except for $T$ which is replaced by $K=\left[\begin{array}{cc}30 & 8 \\ 8 & 10\end{array}\right]$ for aquifer thickness $B=50 \mathrm{~m}$, what will be the discharge vector?

Draw the discharge vector and grad ( $\Phi$ ) on your figure for (i).

Hint,

$$
q=-T\left[\frac{d h}{d x} i+\frac{d h}{d y} j\right], q=\left[\begin{array}{l}
q_{x} \\
q_{y}
\end{array}\right]=-B\left[\begin{array}{ll}
K_{x x} & K_{x y} \\
K_{y x} & K_{y y}
\end{array}\right]\left[\begin{array}{l}
\frac{d h}{d x} \\
\frac{d h}{d y}
\end{array}\right]
$$

## Answer 2.9

(i)

(ii) $\quad q=-T\left[\frac{d h}{d x} i+\frac{d h}{d y} j\right]$

$$
\begin{aligned}
& q_{x}=-T \frac{\partial h}{\partial x}=-5000 \times \frac{12.1-15}{200-0}=72.5 \mathrm{~m}^{2} / \text { day } \\
& q_{y}=-T \frac{\partial h}{\partial y}=-5000 \times \frac{10.4-15}{300-0}=76.65 \mathrm{~m}^{2} / \text { day } \\
& q=\sqrt{(72.5)^{2}+(76.65)^{2}}=105.5 \mathrm{~m}^{2} / \text { day } \\
& \theta=\tan ^{-1}\left(\frac{76.65}{72.5}\right) \Rightarrow \theta=46.6^{\circ} \quad \text { from east }
\end{aligned}
$$

(iii)

$$
\begin{aligned}
& q_{x}=-B\left[k_{x x} \frac{\partial h}{\partial x}+k_{x y} \frac{\partial h}{\partial y}\right]=-50 \times\left[30 \times \frac{-2.9}{200}+8 \times \frac{-4.6}{300}\right]=27.885 \mathrm{~m}^{2} / \text { day } \\
& q_{y}=-B\left[k_{y x} \frac{\partial h}{\partial x}+k_{y y} \frac{\partial h}{\partial y}\right]=-50 \times\left[8 \times \frac{-2.9}{200}+10 \times \frac{-4.6}{300}\right]=13.467 \mathrm{~m}^{2} / \text { day } \\
& q=\sqrt{(27.885)^{2}+(13.467)^{2}}=30.97 \mathrm{~m}^{2} / \text { day } \\
& \theta=\tan ^{-1}\left(\frac{13.467}{27.885}\right) \Rightarrow \theta=25.78^{\circ} \text { from east }
\end{aligned}
$$



## Example 2.10: Groundwater Flow Equations

Let $K_{x}=26 \mathrm{~m} /$ day and $K_{y}=16 \mathrm{~m} /$ day be the principal values of $K$ in an anisotropic aquifer, in the $x$ and $y$ directions respectively for two dimensional flow. The hydraulic gradient is 0.004 in a direction making an angle $30^{\circ}$ with the $+x$ axis.
(i) Determine q
(ii) What is the angle between grad ( $\Phi$ ) and q.

Hint: $\quad q=\left[\begin{array}{l}q_{x} \\ q_{y}\end{array}\right]=-\left[\begin{array}{ll}K_{x} & 0 \\ 0 & K_{y}\end{array}\right]\left[\begin{array}{l}\frac{d \Phi}{d x} \\ \frac{d \Phi}{d y}\end{array}\right]$

## Answer 2.10

$$
\begin{gathered}
\frac{d \Phi}{d x}=\nabla \Phi . i=|\nabla \Phi| \cos \theta=0.004 \cdot \cos \left(30^{\circ}\right)=3.464 \times 10^{-3} \\
\frac{d \Phi}{d y}=\nabla \Phi . j=|\nabla \Phi| \cos (90-\theta)=0.004 \cdot \sin \left(30^{\circ}\right)=2.000 \times 10^{-3} \\
q=\left[\begin{array}{l}
q_{x} \\
q_{y}
\end{array}\right]=-\left[\begin{array}{rr}
36 & 0 \\
0 & 16
\end{array}\right]\left[\begin{array}{l}
3.464 \times 10^{-3} \\
2.000 \times 10^{-3}
\end{array}\right] \\
q=\left[\begin{array}{l}
q_{x} \\
q_{y}
\end{array}\right]=-\left[\begin{array}{c}
36 \times 3.464 \times 10^{-3} \\
16 \times 2.000 \times 10^{-3}
\end{array}\right]=-\left[\begin{array}{l}
0.125 \\
0.032
\end{array}\right] \\
q=\sqrt{(0.125)^{2}+(0.032)^{2}}=0.129 \text { m / day } \\
\theta
\end{gathered}
$$

## Example 2.11: Water Table Map

Figure 2.21 is a map showing the groundwater elevation in wells screened in an unconfined aquifer at Milwaukee, Wisconsin. The aquifer is in good hydraulic connection with Lake Michigan, which has a surface elevation of 580 ft above sea level. Lakes and streams are also shown on the map.
(i) Make a water-table map with a contour interval of 50 ft , starting at 550 ft .
(iv) Why do you suppose that groundwater levels are below the Lake Michigan surface elevation in part of the area?


Figure 2.21 Base map for example 2.11

## Answer 2.11

(i) Here is the Water table map,

(ii) Water levels are below Lake Michigan because of pumping from the aquifer.

## Example 2.12: Steady Flow in a Confined Aquifer

A river and a drainage channel are shown in Figure 2.22. The average elevation of the water surface in the river is 144.6 meters, and in the channel 142.20 meters. The hydraulic conductivity of the confined inter-granular aquifer developed in medium alluvial sand is $3.5 \times 10^{-4}$ $\mathrm{m} / \mathrm{s}$. The hydrologic cross section in Figure 2.23 shows the relationship between the aquifer, the overlying silty clay (aquitard) and the underlying dense (impermeable) clay.


Figure 2.22 Plan view of the river and the drainage channel


Figure 2.23 Hydrogeological cross-section between the river and the drainage channel shown in Figure 11 as determined by filed investigations.
(i) Calculate the groundwater flow per unit width between the river and the drainage channel?
(ii) What is the height $z$ of the piezometric surface at a midpoint between the river and the channel?

## Answer 2.12

(i) According to Darcy law $Q=-$ K.A. $\frac{\Delta h}{\Delta x}=-$ K.b.w. $\frac{\Delta h}{\Delta x}$

So, $q=-K . b . \frac{\Delta h}{\Delta x}$
$q=-K . b . \frac{h_{2}-h_{1}}{L}$
but, $b=\frac{(137.30-133.80)+(139.00-135.40)}{2}=\frac{3.5+3.6}{2}=3.55 \mathrm{~m}$
$\Rightarrow q=-\frac{\left(3.5 \times 10^{-4}\right)(3.55)(142.20-144.5)}{720}$
$\Rightarrow q=4.14 \times 10^{-6} \mathrm{~m}^{2} / \mathrm{s}$
(ii) The position of the potentiometric surface at a midpoint between the river and the channel. Since the aquifer is confined, the piezometric surface is linear.


$$
\begin{aligned}
& q=-K . b . \frac{h_{2}-h_{1}}{x} \\
& \Rightarrow q=-K . b \frac{h_{x}-h_{1}}{x} \\
& \Rightarrow h_{x}=h_{1}-\frac{q}{K b} x \\
& \Rightarrow h_{x}=144.6-\left(\frac{4.14 \times 10^{-6}}{3.5 \times 10^{-4} \times 3.55} \times 3.55\right) \\
& \Rightarrow h_{x}=143.4 \mathrm{masl}
\end{aligned}
$$

Example 2.13: Steady Flow in an Unconfined Aquifer
Refer to Figure 2.24; the hydraulic conductivity of the aquifer is $1.2 \mathrm{~m} /$ day. The value of $h_{1}$ is 17 m and the value of $h_{2}$ is 12 m . The distance from $h_{1}$ to $h_{2}$ is 4525 m . There is an average rate of recharge of $0.0002 \mathrm{~m} /$ day.
(i) What is the average discharge per unit width at $x=0 \mathrm{~m}$ ?
(ii) What is the average discharge per unit width at $x=4252 \mathrm{~m}$ ?
(iii) Where is the water divide located?
(iv) What is the maximum height of the water table?


Figure 2.24

## Answer 2.13

(i) at $x=0$, Use equation 2.111
$q_{x}=\frac{K\left(h_{1}^{2}-h_{2}^{2}\right)}{2 L}-w\left(\frac{L}{2}-x\right)$
$q_{0}=\frac{1.2 \mathrm{~m} / \mathrm{d}\left(17^{2} \mathrm{~m}^{2}-12^{2} \mathrm{~m}^{2}\right)}{2(4525) \mathrm{m}}-0.0002 \mathrm{~m} / \mathrm{d}\left(\frac{4525 \mathrm{~m}}{2}-0\right)$
$q_{0}=-0.43 \mathrm{~m}^{2} /$ day (negative sign implies movement in negative x direction)
(ii) at $x=4525$, Use equation 2.111

$$
\begin{aligned}
& q_{x}=\frac{K\left(h_{1}^{2}-h_{2}^{2}\right)}{2 L}-w\left(\frac{L}{2}-x\right) \\
& q_{4525}=\frac{1.2 \mathrm{~m} / \mathrm{d}\left(17^{2} \mathrm{~m}^{2}-12^{2} \mathrm{~m}^{2}\right)}{2(4525) \mathrm{m}}-0.0002 \mathrm{~m} / \mathrm{d}\left(\frac{4525 \mathrm{~m}}{2}-4525\right) \\
& q_{4525}=0.47 \mathrm{~m}^{2} / \text { day }
\end{aligned}
$$

(iii) From equation 2.112,

$$
\begin{aligned}
& d=\frac{L}{2}-\frac{K}{w} \frac{\left(h_{1}^{2}-h_{2}^{2}\right)}{2 L} \\
& d=\frac{4525 m}{2}-\frac{1.2 m / d}{0.0002 m / d} \frac{\left(17^{2} m^{2}-12^{2} m^{2}\right.}{2(4525)} \\
& d=2166 \text { meters. }
\end{aligned}
$$

(iv) From equation 2.113,

$$
\begin{aligned}
& h_{\max }=\sqrt{h_{1}^{2}-\frac{\left(h_{1}^{2}-h_{2}^{2}\right) d}{L}+\frac{w}{K}(L-d) d} \\
& h_{\max }=\sqrt{17^{2} m^{2}-\frac{\left(17^{2} m^{2}-12^{2} m^{2}\right) \times 2166 m}{4525}+\frac{0.0002 m / d}{1.2 m / d}(4525 m-2166 m) \times 2166 m} \\
& h_{\max }=32.7 m
\end{aligned}
$$

## Example 2.14: The Potentiometric Surfaces for the Steady-State One-Dimensional Groundwater Flow System

(i) Sketch in, as accurately as possible, the potentiometric surfaces for the steadystate one-dimensional groundwater flow systems given below.


Figure 2.25
(ii) For the following hydraulic parameter values determine the distribution of $h(x)$ over the length of the aquifer in Figure 2.25 (b).

$$
\begin{aligned}
& K=10 \mathrm{~m} / \text { day } \\
& h_{1}=10 \mathrm{~m} \\
& h_{2}=5 \mathrm{~m}
\end{aligned}
$$

Determine the largest abstraction that can be obtained theoretically from the aquifer assuming the equation you derive is valid.

## Answer 2.14

(i) (a)
a)

(b) There are two possibilities:
(1) All $Q$ goes to the left hand side as this side has less head.
(2) $Q$ is divided according to length.
b)

(c) There are three possibilities:
(1) $\quad Q_{1}$ as well as $Q_{2}$ is relatively small.
(2) $\quad Q_{1}$ and $Q_{2}$ are big, they seem equal.
(3) $\quad Q_{2}>Q_{1}$

(ii)
b)


Apply Dupuit Assumption, see Equation 2.98

$$
q=\frac{1}{2} K\left(\frac{h_{1}^{2}-h_{2}^{2}}{L}\right)
$$

You can follow the derivation of this equation from Darcy's Law, as shown below.

$$
\begin{gathered}
q=-K h \frac{d h}{d x} \Rightarrow \quad \int_{0}^{L} q d x=-K \int_{h_{1}}^{h_{2}} h d h \Rightarrow \quad q L=-K\left(\frac{h_{2}^{2}}{2}-\frac{h_{1}^{2}}{2}\right) \\
\text { Then, } \quad q=\frac{1}{2} K\left(\frac{h_{1}^{2}-h_{2}^{2}}{L}\right)
\end{gathered}
$$

Now, we can calculate the total flow in the aquifer,

$$
q_{\text {Total }}=\frac{1}{2} \times 10\left(\frac{100-25}{120}\right)=3.125 \mathrm{~m}^{3} / \text { day } / \mathrm{m}
$$

Now, $Q=q_{1}+q_{2}$

$$
\begin{gathered}
q_{1}=\frac{10}{2}\left[\frac{10^{2}-h_{o}^{2}}{70}\right]=\frac{5}{70}\left[100-h_{o}^{2}\right] \\
q_{2}=\frac{10}{2}\left[\frac{5^{2}-h_{o}^{2}}{50}\right]=\frac{5}{50}\left[25-h_{o}^{2}\right] \\
Q=\frac{5}{70} \times 100-\frac{5}{70} h_{o}^{2}+\frac{5}{50} \times 25-\frac{5}{50} h_{o}^{2} \\
Q=\frac{50}{7}-\frac{5}{70} h_{o}^{2}+\frac{5}{2}-\frac{1}{10} h_{o}^{2} \\
\frac{d Q}{d h_{o}}=-\frac{10}{70} h_{o}-\frac{2}{10} h_{o}=0 \\
\Rightarrow \frac{24}{70} h_{o}=0 \Rightarrow h_{o}=0 \\
\Rightarrow \quad Q=9.64 \mathrm{~m}^{3} / \text { day/m width }
\end{gathered}
$$

Example 2.15: The Potentiometric Surfaces for the Steady-State One-Dimensional Groundwater Flow System

Sketch in, as accurately as possible, the potentiometric surfaces for the steady-state onedimensional groundwater flow systems given below.


Figure 2.26
(a) Recharge $=q L$ and Discharge $=q L / 2$

(b)
(9) $\mathrm{q} /$ unit length $\longrightarrow \mathrm{qL} / 2<\mathrm{Q}<\mathrm{qL}$

(c)


Example 2.16: The Potentiometric Surfaces for the Steady-State One-Dimensional Groundwater Floe System

Sketch in, as accurately as possible, the potentiometric surfaces for the steady-state onedimensional groundwater flow systems given below.
(a)

(b)


Figure 2.27

For Figure 2.27 b, assume that

$$
\begin{aligned}
Q_{1} & =-\frac{B K\left(h_{2}-h_{1}\right)}{L} \\
Q_{2} & =+\frac{B K\left(h_{2}-h_{1}\right)}{L}
\end{aligned}
$$

## Answer 2.16

(a) You need to locate $h_{0}$ at the abstraction line as $Q=q_{1}+q_{2}$

(b) $\quad Q_{1}=q_{1}+q_{2}$ and $Q_{2}=-q_{2}+q_{3}$


## First

$$
\begin{align*}
Q_{1} & = \\
\frac{-B K\left(h_{2}-h_{1}\right)}{L} & =\frac{3 B K\left(h_{1}-h_{o}\right)}{L}+\frac{q_{2}}{2 B K\left(h_{3}-h_{0}\right)} \\
L & -h_{2}+h_{1} \\
\Rightarrow \quad & =3 h_{1}-3 h_{o}+3 h_{3}-3 h_{0}  \tag{1}\\
\Rightarrow \quad 6 h_{0} & =3 h_{1}+h_{2}+3 h_{3}
\end{align*}
$$

## Second

$$
\begin{align*}
Q_{2} & = \\
\frac{B K\left(h_{2}-h_{1}\right)}{L} & =\frac{3 B K\left(q_{2}-h_{3}\right)}{L}+\frac{3 B K\left(h_{2}-h_{3}\right)}{L} \\
\Rightarrow \quad h_{2}-h_{1} & =3 h_{0}-3 h_{3}+3 h_{2}-3 h_{3} \\
\Rightarrow \quad-3 h_{0} & =h_{1}+2 h_{2}-6 h_{3} \tag{2}
\end{align*}
$$

Multiply [(1)×2] and add (1) to (2)

$$
\begin{aligned}
12 h_{0} & =4 h_{1}+2 h_{2}+6 h_{3} \\
+ & =h_{1}+2 h_{2}-6 h_{3} \\
-3 h_{0} & =5 h_{1}+4 h_{2} \\
\frac{9 h_{0}}{} & =\frac{5}{9} h_{1}+\frac{4}{9} h_{2}
\end{aligned}
$$

## Solving for $h_{\mathbf{3}}$, substitute $h_{\mathbf{o}}$ in (2)

$$
\begin{aligned}
-3 h_{0} & =h_{1}+2 h_{2}-6 h_{3} \\
\frac{5}{9} h_{1}+\frac{4}{9} h_{2} & =h_{1}+2 h_{2}-6 h_{3} \\
\Rightarrow \quad 6 h_{3} & =\left(h_{1}-\frac{5}{9} h_{1}\right)+\left(2 h_{2}-\frac{4}{9} h_{2}\right)=\left(\frac{4}{9} h_{1}\right)+\left(\frac{14}{9} h_{2}\right) \\
\Rightarrow \quad h_{3} & =\frac{4}{54} h_{1}+\frac{14}{54} h_{2}
\end{aligned}
$$

## Example 2.17: One-Dimensional Flow Regime

For the one-dimensional flow regime shown in Figure 2.28,
(i) Calculate how long it will take a particle to travel from point $A$ to the river $B$ assuming that it travels with the averaged advective velocity of the groundwater flow.
(ii) Sketch in, as accurately as possible, the potentiometric surface for the steady-state one-dimensional groundwater flow systems given in the Figure.


Figure 2.28 (Not to scale)

## Answer 2.17

(i)

From the continuity equation $q_{1}=q_{2}=q_{3}$,

$$
\begin{array}{rlrl}
q_{1} & =q_{2} \\
& \frac{-K\left(h_{1}-h_{A}\right)}{L} & =\frac{-K\left(h_{2}-h_{1}\right)}{L} \\
\Rightarrow \quad 10\left(h_{1}-35\right) & =13\left(h_{2}-h_{1}\right) \\
\Rightarrow \quad 10 h_{1}-350 & =13 h_{2}-13 h_{1} \\
\Rightarrow \quad 23 h_{1} & =13 h_{2}+350 \tag{1}
\end{array} \quad \text { (Note } L \text { is the same) }
$$

$$
\begin{array}{rlrl} 
& q_{1} & =q_{3} \\
& \frac{-K\left(h_{1}-h_{A}\right)}{L} & =\frac{-K\left(h_{B}-h_{2}\right)}{L} & \\
\Rightarrow \quad 10\left(h_{1}-35\right) & =15\left(10-h_{2}\right) & \\
\Rightarrow \quad 10 h_{1}-350 & =150-15 h_{2} & \\
\Rightarrow \quad 10 h_{1} & =-15 h_{2}+500 & \\
\Rightarrow \quad & 2 h_{1} & =-3 h_{2}+100 & \tag{2}
\end{array}
$$

## Multiply [(1) $\times 3$ ] and [(2)× 13] and add them

$$
\begin{aligned}
& 69 h_{1}=39 h_{2}+1,050 \\
&+ \\
& 26 h_{1}=-39 h_{2}+1300
\end{aligned}
$$

$$
\begin{aligned}
95 h_{1} & =2,350 \\
\Rightarrow h_{1} & =24.74 \mathrm{~m}
\end{aligned}
$$

Solving for $h_{2}$, substitute $h_{1}$ in (1)

$$
\begin{aligned}
23 h_{1} & =13 h_{2}+350 \Rightarrow(23 \times 24.74)=13 h_{2}+350 \\
\Rightarrow \quad h_{2} & =16.85 \mathrm{~m}
\end{aligned}
$$

## Now,

$$
\begin{aligned}
& u_{1}=-\frac{K_{1}}{n_{1}} \times \frac{\left(h_{1}-h_{A}\right)}{L_{1}}=-\frac{10}{0.2} \times \frac{(24.74-35)}{300}=1.71 \mathrm{~m} / \text { day } \\
& \Rightarrow t_{1}=\frac{L_{1}}{u_{1}}=\frac{300 \mathrm{~m}}{1.17 \mathrm{~m} / \text { day }}=175.44 \text { days } \\
& u_{2}=-\frac{K_{2}}{n_{2}} \times \frac{\left(h_{2}-h_{1}\right)}{L_{2}}=-\frac{13}{0.25} \times \frac{(16.85-24.74)}{300}=1.37 \mathrm{~m} / \text { day } \\
& \Rightarrow t_{2}=\frac{L_{2}}{u_{2}}=\frac{300 \mathrm{~m}}{1.37 \mathrm{~m} / \text { day }}=218.98 \text { days } \\
& u_{3}=-\frac{K_{3}}{n_{3}} \times \frac{\left(h_{B}-h_{2}\right)}{L_{3}}=-\frac{15}{0.27} \times \frac{(10-16.85)}{300}=1.27 \mathrm{~m} / \text { day } \\
& \Rightarrow t_{3}=\frac{L_{3}}{u_{3}}=\frac{300 \mathrm{~m}}{1.27 \mathrm{~m} / \text { day }}=236.22 \text { days }
\end{aligned}
$$

It will take a particle to travel to travel from point $A$ to the river $B$ about 631 days
(ii) Note that the sketch is not to scale.


## Example 2.18: Flow Net

Figure 2.29 is a piezometric map of a shallow unconfined groundwater catchment running down to the sea.
(i) Sketch streamlines on this map in order to form a flow net.
(ii) Estimate the flow rate of groundwater across the line on the map indicated by the 20 m contour, assuming an aquifer transmissivity of $250 \mathrm{~m}^{2} /$ day.


Figure 2.29

## Answer 2.18

(i)

(ii) $\quad Q=T \cdot \frac{d h}{d l}$. width

* From Figure 2.29, it is approximated (measured) that distance between contour 80 and contour 20 is about 5 cm , i.e. 0.05 m .
It is also approximated (measured) that the width of contour 20 is $10 \mathrm{~cm}(0.1 \mathrm{~m})$.
Note that the scale is $1: 25,000$.

$$
\begin{aligned}
Q & =T \cdot \frac{d h}{d l} \cdot \text { width } \\
\Rightarrow Q & =250 \times\left[\frac{80-20}{0.05 \times 25,000}\right] \times[0.1 \times 25,000] \\
Q & =30,000 \mathrm{~m}^{3} / \text { day }
\end{aligned}
$$

## Example 2.19: Flow Nets and Well Operation

(i) The attached map (Figure 2.30) gives the positions of seven observation boreholes in a limestone aquifer, with piezometric levels for each borehole in meters above sea level. The mean stage of the river is 65 m above sea level. Construct a flow net representing the equipotential lines and selected flow lines in the aquifer.
(ii) Pumping tests in wells 2 and 6 have yielded transmissivity values of $110 \mathrm{~m}^{2} /$ day and $95 \mathrm{~m}^{2} /$ day respectively. The effective saturated thickness of the aquifer in the area is 45 m . Use Darcy's Law to calculate the flow of groundwater into the river when the potentiometric surface is in the configuration you have sketched on Figure 2.29.


Figure 2.30 Piezometric levels (meters above sea level [masl]) in the Sherburn Limestone Aquifer, 1-7-1992
(i)

(ii)

From the given scale: $1 \mathrm{~km}=1.7 \mathrm{~cm}$.
So, the distance between well No. 2 and well No. $6=10.5 \mathrm{~cm}=6.176 \mathrm{~km}$
And the length of the segment of the river that faces the groundwater flow $=11.2 \mathrm{~cm}=$ 6.588 km .

$$
\begin{aligned}
Q & =T \cdot \frac{d h}{d l} \cdot \text { width } \\
\Rightarrow Q & =\left(\frac{110+95}{2}\right) \times\left[\frac{93.4-65}{6176}\right] \times[6588] \\
Q & =3,105.19 \mathrm{~m}^{3} / \text { day }
\end{aligned}
$$

## Example 2.20: Analysis of Groundwater Flow Systems

## INTRODUCTION

Analysis of groundwater flow system involves identification of groundwater flow directions and qualification of fluxes. The main method for achieving these objectives is the construction of flow nets. A flow net is an assemblage of contours of groundwater head, with flow-lines drawn at right angles to the contours. It is usual to construct flow nets in two orientations, i.e. a plan view [water table map] and one or more vertical profiles [usually cross sections along the trace of a flow line identified on the water table map].

## MA TERIALS

You are provided with a map showing groundwater levels (in meters above mean sea level) on a single day in July 1990 in an extensive limestone aquifer (lithologically similar to the deep West Bank aquifer). You should bring drawing equipment (rulers, protractors, set-squares, coloured pencils etc.). (Should you complete the manual steps of this exercise, software is available for step 8).

## What You have to do

1 Draw water table contours on the map provided. Pay particular attention to the need to represent cones of depression around abstraction boreholes (i.e. pumping wells). [Hint: the coast-line is fixed head boundary of 0 meter water table elevation].
2 Compute the flow net in plan by adding ten flow lines, perpendicular to the contours.
3 Examine the shape of your flow net:
a. Does the pattern of contours (spacing, orientation) tell you anything about local variations in aquifer permeability?
b. How many distinct discharge areas can you recognize?
c. What is the hydrogeological influence of the fault at the southern edge of the mapped area?

4 Given a mean transmissivity of $250 \mathrm{~m}^{2} /$ day, what volume of water flows daily from the aquifer into the ocean? Is the flow rate constant along the coastline, or does it vary from one stretch to another?
5 Given that the abstraction boreholes shown on the map are long-established (with steady drawdowns), and given their combined abstraction rate is around $100,000 \mathrm{~m}^{3} /$ day, estimate the amount of recharge this aquifer is receiving.
6 Choose one of the flow lines you have drawn on the map, and for each construct a vertical profile flow net, showing the inferred distribution of groundwater head in the subsurface. You may assume that, from its outcrop position, the base of the aquifer dips eastwards at a gradient of 25 m per kilometer.
7 Assuming a hydraulic conductivity of $25 \mathrm{~m} /$ day is valid for both profiles, what would be the average transit time to the coast for a molecule of water entering the aquifer near the outcrop of the base of the limestone? Does the result surprise you? If so, why?


Features which give groundwater level data

## LEGEND

- Observation borhole
- Abstraction borhole

A Spring


Figure 2.31
(1) and (2)

a. Closer contour lines indicate lower hydraulic gradient, hence lower permeability.
b. Nearly 2.
c. It acts as a no-flow boundary.

4

$$
\begin{aligned}
& Q=T \cdot \frac{d h}{d l} \cdot \text { width } \\
Q & =Q_{1}(\text { discharg e area } 1)+Q_{2}(\text { discharge area } 2) \\
\Rightarrow & Q_{1}=250 \times \frac{(42-0)}{6000} \times 10,000=17,500 \mathrm{~m}^{3} / \text { day } \\
\Rightarrow Q_{2} & =250 \times \frac{(70-0)}{15,000} \times 16,000=18,667 \mathrm{~m}^{3} / \text { day } \\
& Q=17,500+18667=36,167 \mathrm{~m}^{3} / \text { day } .
\end{aligned}
$$

The flow rate along the coastline varies from one stretch to another.

5 The Abstraction $=100,000 \mathrm{~m}^{3} /$ day
Flow to the ocean $=36,167 \mathrm{~m}^{3} /$ day
So, theoretically, the amount of recharge that this aquifer is receiving $=136,167 \mathrm{~m}^{3} /$ day .
6
The vertical profile flow net for the flow line marked by (1).


7

$$
\begin{aligned}
u & =\frac{Q}{n}=K \times i=25 \mathrm{~m} / \text { day } \times \frac{50}{10,000} \\
u & =0.125 \mathrm{~m} / \text { day } . \\
t & =\frac{10,000}{0.125}=80,000 \text { days }=219 \text { years. } .
\end{aligned}
$$

## Example 2.21 Application of Darcy's Law in Landfills

A landfill liner (very low permeable material) is laid on a top of permeable sand unit while the sides of the liner have very contacts with the sand unit and a good clay unit as shown in Figure 2.32. The landfill can be represented by a square with length of 150 m on a side and a vertical depth of 15 m from the surface. Thickness of landfill clay liner $=0.45 \mathrm{~m}$. Hydraulic conductivity of landfill clay liner $=8.64 \times 10^{-4} \mathrm{~m} /$ day. The regional groundwater level for the confined sand is located 3 m below the surface. How much water will have to be continuously pumped from the landfill to keep the potentiometric surface at 18 m above sea level within the landfill?
(Hint: Assume horizontal flow and use Darcy's Law only)


Figure 2.32

> Flow into landfill = flow to be pumped

Since flow is horizontal, Darcy's law can be applied

$$
Q=K \cdot A \cdot \frac{\Delta h}{\Delta l}
$$

$\checkmark \quad$ Flow to landfill will come form the sides of the landfill that have contacts with sand + from the bottom of the landfill because it also has contacts with landfill.
$\checkmark \quad$ To apply Darcy's law for horizontal flow, means that head is constant vertically, then hydraulic gradient for the given situation is

$$
i=\frac{\Delta h}{\Delta l}=\frac{27-18}{0.45}=20
$$

Note that the thickness of the clay liner $=0.45 \mathrm{~m}$
$\checkmark \quad$ Now, we have four sides that the landfill has contact with them. These four sides are identical:

Area of each side $=(22.5-15) \times 150=1125 m^{2}$
Total area of the four sides $\left(A_{s}\right)=4 \times 1125=4500 \mathrm{~m}^{2}$
Flow into landfill from sides $Q_{s}=$ K.A. $\frac{\Delta h}{\Delta l}$

$$
\begin{aligned}
& =8.64 \times 10^{-4} \frac{\mathrm{~m}}{\text { day }} \times 4500 \mathrm{~m}^{2} \times 20 \\
Q_{s} & =78 \mathrm{~m}^{3} / \text { day }
\end{aligned}
$$

$\checkmark \quad$ Now, we have one bottom that has contacts with the landfill;
Area of the bottom $\left(A_{b}\right)=150 \times 150=22,500 \mathrm{~m}^{2}$
Flow int o landfill from bottom $Q_{b}=$ K.A. $\frac{\Delta h}{\Delta l}$

$$
\begin{aligned}
& =8.64 \times 10^{-4} \frac{\mathrm{~m}}{d a y} \times 22,500 \mathrm{~m}^{2} \times 20 \\
Q_{b} & =389 \mathrm{~m}^{3} / \text { day }
\end{aligned}
$$

$\checkmark \quad$ The flow into landfill $=Q_{s}+Q_{b}=78+389=467 \mathrm{~m}^{3} /$ day, which is the same as volume of water that should be pumped to keep the landfill hydraulics as shown in the figure.

## Example 2.22

A sand aquifer 12.19 m thick is about 1.61 km wide. The aquifer is covered by a confining unit of glacial till about 13.72 m thick beginning from the land surface. The difference in the hydraulic head between two wells 1524 m apart is 3.05 m (Fig. 6.4). The hydraulic conductivity of the sand aquifer is $20.44 \mathrm{~m} /$ day. What is the quantity of ground water passing through a cross-section of this aquifer in $\mathrm{ft}^{3} /$ day, gallons per day, and $\mathrm{m}^{3} /$ day?

## Method 1:

$$
\begin{align*}
& \mathrm{A}=(12.19 \mathrm{~m})(1610 \mathrm{~m})=19626 \mathrm{~m}^{2} \\
& \mathrm{i}=\Delta \mathrm{h} / \mathrm{L}=(3.05 \mathrm{~m}) /(1524 \mathrm{~m})=0.002 \\
& \mathrm{~K}=20.44 \mathrm{~m} / \text { day } \\
& \mathrm{Q}=\mathrm{KAi}=\left(20.44 \frac{\mathrm{~m}}{\text { day }}\right)\left(19626 \mathrm{~m}^{2}\right)(0.002)=802 \frac{\mathrm{~m}^{3}}{\text { day }} \\
& \mathrm{Q}=802 \frac{\mathrm{~m}^{3}}{\text { day }} \times 264=211,728 \mathrm{gpd} \\
& \mathrm{Q}=802 \frac{\mathrm{~m}^{3}}{\text { day }} \times 35.32=28,327 \frac{\mathrm{ft}^{3}}{\text { day }}
\end{align*}
$$

## Method 2:

$$
\begin{align*}
& \mathrm{i}=\Delta \mathrm{h} / \mathrm{L}=(3.05 \mathrm{~m}) /(1524 \mathrm{~m})=0.002 \\
& \mathrm{~K}=20.44 \mathrm{~m} / \text { day } \\
& \mathrm{b}=12.19 \mathrm{~m} \\
& \mathrm{~W}=161 \mathrm{~km}=161,000 \mathrm{~m} \\
& \mathrm{~T}=\mathrm{Kb}=(20.44 \mathrm{~m} / \text { day })(12.19 \mathrm{~m})=249 \mathrm{~m}^{2} / \text { day } \\
& \mathrm{Q}=\mathrm{TWi}=\left(249 \frac{\mathrm{~m}^{2}}{\text { day }}\right)(1,610 \mathrm{~m})(0.002)=802 \frac{\mathrm{~m}^{3}}{\text { day }}
\end{align*}
$$



Figure 6.4.

## Example 2.23

A confining unit comprised of silt is 3.65 meters thick. The vertical hydraulic conductivity of this unit is $0.085 \mathrm{~m} /$ day. The $\Delta \mathrm{h}$ between wells tapping the upper and lower confined aquifers is 4.60 meters. What is the volume of water leaking from the deeper aquifer to the shallow aquifer (Fig. 6.5). The area under influence of leakage is $1.3 \mathrm{~km}^{2}$.


Figure 6.5.

Modify Darcy's Law to determine the volume of water supplied by aquifer leakage:

$$
\mathrm{Q}_{\mathrm{L}}=\left(\frac{\mathrm{K}_{\mathrm{v}}}{\mathrm{~b}_{\mathrm{L}}}\right)(\mathrm{A})(\Delta \mathrm{h})
$$

where, for consistent units,
$\mathrm{K}_{\mathrm{v}}=$ vertical hydraulic conductivity of the leaky unit $=\mathrm{L} / \mathrm{T}$
$\mathrm{b}_{\mathrm{L}}=$ thickness of the confining unit $=\mathrm{L}$
$\mathrm{K}_{\mathrm{v}}=0.085 \mathrm{~m} /$ day
$\mathrm{b}_{\mathrm{L}}=3.65 \mathrm{~m}$
$\mathrm{A}=1.3 \mathrm{~km}^{2}=1,300,000 \mathrm{~m}^{2}$
$\Delta \mathrm{h}=4.60 \mathrm{~m}$
$Q_{L}=\left(\frac{0.085 \frac{\mathrm{~m}}{\text { day }}}{3.65 \mathrm{~m}}\right)\left(1,300,000 \mathrm{~m}^{2}\right)(4.60 \mathrm{~m})=139,260 \frac{\mathrm{~m}^{3}}{\text { day }}$
$Q_{L}=36,764,640 \mathrm{gpd}$
$\mathrm{Q}_{\mathrm{L}}=4,918,663 \mathrm{ft}^{3} /$ day

## Example 2.24

The radius of a production well and gravel pack is $(12.70 \mathrm{~cm}+22.86 \mathrm{~cm}=35.56 \mathrm{~cm})$. The length of the well screen is 27.43 m . The production well discharge is $9.0 \mathrm{~m}^{3} / \mathrm{min}$. The intrinsic permeability of the gravel pack is $4.5 \times 10^{-5} \mathrm{~cm}^{2}$. Assess the validity of Darcy's Law for flow near the well screen at a ground-water temperature of $15^{\circ} \mathrm{C}$. The mean grain diameter of the gravel pack is $5.0 \mathrm{~mm}(=0.50 \mathrm{~cm})$.

The area of a cylinder $=2 \pi \mathrm{rh}$
where, for consistent units,

$$
\begin{aligned}
\mathrm{r} & =\text { radius }=\mathrm{L} \\
\mathrm{~h} & =\text { height of the cylinder }=\mathrm{L} .
\end{aligned}
$$

The specific discharge ( $\mathbf{v}_{\mathbf{s}}$ ) for the area of a cylinder represented by the radius of the well + gravel pack and the length of the well screen is

$$
\begin{aligned}
& \mathrm{v}_{\mathrm{s}}=\frac{\mathrm{Q}}{\mathrm{~A}}=\frac{\mathrm{Q}}{2 \pi \mathrm{rh}} \\
& \mathrm{v}_{\mathrm{s}}=9 \mathrm{~m}^{3} / \mathrm{min}=0.15 \mathrm{~m}^{3} / \mathrm{s} \\
& \mathrm{r}=35.56 \mathrm{~cm}=0.3556 \mathrm{~m} \\
& \mathrm{~h}=27.43 \mathrm{~m} \\
& \mathrm{v}=\mathrm{v}_{\mathrm{s}}=\frac{\mathrm{Q}}{\mathrm{~A}}=\frac{\mathrm{Q}}{2 \pi \mathrm{rh}}=\frac{0.15 \frac{\mathrm{~m}^{3}}{\mathrm{~s}}}{(2 \pi)(0.3556 \mathrm{~m})(27.43 \mathrm{~m})}=0.00245 \mathrm{~m} / \mathrm{s}=0.245 \mathrm{~cm} / \mathrm{s}
\end{aligned}
$$

Compute the Reynolds number using Table 6.2 and equation (6.8).

$$
\begin{align*}
& \rho_{\mathrm{at} 15^{\circ} \mathrm{C}}=0.9991 \mathrm{~g} / \mathrm{cm}^{3} \\
& \mu_{\mathrm{at} 15^{\circ} \mathrm{C}}=0.0114 \mathrm{~g} / \mathrm{s} \cdot \mathrm{~cm} \\
& \mathrm{R}_{\mathrm{N}}=\frac{\rho \mathrm{vd}}{\mu}=\frac{\left(0.9991 \frac{\mathrm{~g}}{\mathrm{~cm}^{3}}\right)\left(0.245 \frac{\mathrm{~cm}}{\mathrm{~s}}\right)(0.50 \mathrm{~cm})}{0.0114 \frac{\mathrm{~g}}{\mathrm{~s} \cdot \mathrm{~cm}}}=10.74
\end{align*}
$$

$R_{N}$ is greater than 10 ; indicating that flow velocity near the well screen is turbulent and that Darcy's Law is not applicable.

## Example 2.25

Make a copy of Figures $8.26,8.27$, and 8.28. Draw sufficient flow lines on them to illustrate the regional flow patterns. Even though these aquifers are anisotropic, make the flow lines cross the equipotentials at right angles.


FIGURE 8.26 Hydrogeologic cross section showing head distribution in a one-lake system with a homogeneous, anisotropic aquifer system. Results are based on a two-dimensional, steady-state, numerical-simulation model. Source: T. C. Winter, U.S. Geological Survey Professional Paper 1001, 1976.


FIGURE 8.27 Hydrogeologic cross section showing head distribution in a one-lake system with a layered aquifer system. The high-conductivity layer has a conductivity 1000 times as great as the low-conductivity layer. The lake loses water to the aquifer. Source: T. C. Winter, U.S. Geological Survey Professional Paper 1001, 1976.


FIGURE 8.28 Hydrogeologic cross section through a three-lake system with a complex aquifer. Local and regional ground-water flow systems are present. Source: T. C. Winter, U.S. Geological Survey Professional Paper 1001, 1976.

## Solution:



FIGURE 8.26 Hydrogeologic cross section showing head distribution in a one-lake system with a homogeneous, anisotropic aquifer system. Results are based on a two-dimensional, steady-state, numerical-simulation model. Source: T. C. Winter, U.S. Geological Survey P;ofessional Paper 1001, 1976.


FIGURE 8.27 Hydrogeologic cross section showing head distribution in a one-lake system with a layered aquifer system. The high-conductivity layer has a conductivity 1000 times as great as the low-conductivity layer. The lake loses water to the aquifer. Source: T. C. Winter, U.S. Geological Survey Professional Paper 1001, 1976.


FIGURE 8.28 Hydrogeologic cross section through a three-lake system with a complex aquifer. Local and regional ground-water flow systems are present. Source: T. C. Winter, U.S. Geological Survey Professional Paper 1001, 1976.

## Example 2.26

Based on the flow fields shown on Figures 8.3 and 8.7, draw a flow net on Figure 8.30.


No flow boundary
FIGURE 8.30 Diagram for Example 2.26


FIGURE 8.3 Regional flow pattern in an area of sloping linear topography and water table. The flow pattern is symmetrical about the midline. Source: J. A. Tóth, Journal of Geophysical Research 67 (1962): 4375-87.


FIGURE 8.7 A. Equipotential field and flow lines in a region where a high-conductivity body is buried in a lower-conductivity aquifer. B. The water table and the potentiometric profile of a line of piezometers, each ending at the same elevation along line $A-A^{\prime}$ of Part $A$.


## Example 2.27

Answer the following questions based on Figure 8.31.
A. Fill in the heads at the locations labeled on the diagram.

Location Elevation Head Pressure Head Total Head
A
$\qquad$
B
$\qquad$

C
$\qquad$
D
$\qquad$
E
$\qquad$
F
$\qquad$
B. Find one place on Figure 8.31 where recharge is occurring, and label it $R$.
C. Find one place on Figure 8.31 where discharge is occurring, and label it $D$.
D. Draw in flow lines on Figure 8.31 starting at points $X, Y$, and $Z$.


FIGURE 8.31 Diagram for Example 2.27. The dashed lines are equipotentials. Heads are in meters.

## Solution:



Note that part $A$. can be answered using another datum. For example, if one uses the bottom of the aquifer as the datum, the elevation and total head changes.
B. Find one place on Figure 8.31 where recharge is occurring, and label it $R$.
C. Find one place on Figure 8.31 where discharge is occurring, and label it $D$.
D. Draw in stream lines on 8.31 starting at points $X, Y$, and $Z$.


FIGURE 8.31 Diagram for Example 2.27. The dashed lines are equipotentials. Heads are in meters.

## Example 2.28

A saline solution with a concentration of $370 \mathrm{mg} / \mathrm{L}$ is introduced into a $2-m$-long sand column in which the pores are initially filled with distilled water. If the solution drains through the column at an average linear velocity of $0.79 \mathrm{~m} /$ day, and the dynamic dispersivity of the sand column is 15 cm , what would the concentration of the effluent be 1.8 days after flow begins?

## Solution:

Using Equation 11-8:
A. Calculate $D_{L}$ from Equation 11-6:

$$
\begin{array}{ll}
\mathrm{D}_{\mathrm{L}} & =\mathrm{a}_{\mathrm{L}} \mathrm{v}+\mathrm{D}^{*} \quad\left(\text { assume } \mathrm{D}^{*}=0\right) \\
\mathrm{D}_{\mathrm{L}} & =(.15 \mathrm{~m})(0.79 \mathrm{~m} / \mathrm{d}) \\
\mathrm{D}_{\mathrm{L}} & =0.119 \mathrm{~m}^{2} / \mathrm{d} \\
\mathrm{v}_{\mathrm{X}} & =0.79 \mathrm{~m} / \mathrm{d} \\
\mathrm{~L} & =2 \mathrm{~m} \\
\mathrm{C}_{\mathrm{o}} & =370 \mathrm{mg} / \mathrm{L} \\
\mathrm{t} & =1.8 \mathrm{~d}
\end{array}
$$

B. Using Equation 11-8:

$$
\begin{gathered}
C=\frac{C_{0}}{2}\left[\operatorname{erfc}\left(\frac{L-v_{x} t}{2 \sqrt{D_{L} t}}\right)+\exp \left(\frac{v_{x} L}{D_{L}}\right) \operatorname{erfc}\left(\frac{L+v_{x} t}{2 \sqrt{D_{L} t}}\right)\right] \\
C=\frac{370 \mathrm{mg}}{2}\left[\operatorname{erfc}\left(\frac{2 m-(0.79 \mathrm{~m} / \mathrm{dx1.8d})}{2 \sqrt{0.119 \frac{\mathrm{~m}^{2}}{d} \times 1.8 \mathrm{~d}}}\right)+\exp \left(\frac{0.79 \mathrm{~m} / \mathrm{d} \times 2 \mathrm{~m}}{0.119 \frac{\mathrm{~m}^{2}}{d}}\right)\right. \\
\left.x \operatorname{erfc}\left(\frac{2 m+(0.79 \mathrm{~m} / \mathrm{dx1.8d})}{2 \sqrt{0.119 \frac{\mathrm{~m}^{2}}{d} \times 1.8 \mathrm{~d}}}\right)\right]
\end{gathered}
$$

```
C = 185[erfc(0.624) + exp(13.28) x erfc(3.70)]
```

since erfc $(3.70) \simeq 0$, ignore second term.

$$
C=185 \mathrm{mg} / \mathrm{L}[\operatorname{erfc}(0.624)]
$$

from Appendix 13 , erfc $(0.624) \cong 0.378$

$$
\begin{aligned}
& C=185 \mathrm{mg} / \mathrm{L}(0.378) \\
& C=70 \mathrm{mg} / \mathrm{L}
\end{aligned}
$$

## Example 2.29

Given the flow situation above, what would the effluent concentration be 2.1 days after flow begins?

## Solution:

At $t=2.1$ days with the same flow situation as problem 11-1:

$$
\begin{aligned}
\mathrm{D}_{\mathrm{L}} & =0.119 \mathrm{~m}^{2} / \mathrm{d} \\
\mathrm{v}_{\mathrm{x}} & =0.079 \mathrm{~m} / \mathrm{d} \\
\mathrm{~L} & =2 \mathrm{~m} \\
\mathrm{C}_{\mathrm{o}} & =370 \mathrm{mg} / \mathrm{L} \\
\mathrm{t} & =2.1 \mathrm{~d}
\end{aligned}
$$

Using Equation 11-8:

$$
\begin{gathered}
c=\frac{370 \mathrm{mg}}{2}\left[\operatorname { e r f c } \left(\frac{2 \mathrm{~m}-(0.79 \mathrm{~m} / \mathrm{d} \times 2.1 \mathrm{~d})}{\left.2 \sqrt{0.119 \frac{\mathrm{~m}^{2}}{\mathrm{~d}} \times 2.1 \mathrm{~d}}\right)+\exp \left(\frac{0.79 \mathrm{~m} / \mathrm{d} \times 2 \mathrm{~m}}{0.119 \frac{\mathrm{~m}^{2}}{\mathrm{~d}}}\right)}\right.\right. \\
\left.\times \operatorname{erfc}\left(\frac{2 \mathrm{~m}+(0.79 \mathrm{~m} / \mathrm{d} \times 2.1 \mathrm{~d})}{2 \sqrt{0.119 \frac{\mathrm{~m}^{2}}{\mathrm{~d}} \times 2.1 \mathrm{~d}}}\right)\right]
\end{gathered}
$$

```
C}=185\frac{\textrm{mg}}{\textrm{L}}[\operatorname{erfc}(0.341)+\operatorname{exp}(13.28) x\operatorname{erfc}(3.66)
```

and since erfc $(3.66) \simeq 0$, ignore second term.

$$
C=185 \mathrm{mg} / \mathrm{L}[\operatorname{erfc}(0.341)]
$$

from Appendix 13, $\operatorname{erfc}(0.341) \cong 0.631$
$\mathrm{C}=185 \mathrm{mg} / \mathrm{L} \times 0.631$
$C=117 \mathrm{mg} / \mathrm{L}$

## Example 2.30

A landfill is leaking an effluent with a concentration of sodium of $1250 \mathrm{mg} / \mathrm{L}$. It seeps into an aquifer with a hydraulic conductivity of $7.3 \mathrm{~m} /$ day, a gradient of 0.0030 and an effective porosity of 0.25 . A downgradient monitoring well is located 25 m from the landfill. What would the sodium concentration be in this monitoring well 300 days after the leak begins? Note: In this problem you will find that you need to find erfc(-x). This is equal to $1+\operatorname{erf}(x)$.

Solution:

$$
\begin{aligned}
\text { Given: } \mathrm{K} & =7.3 \mathrm{~m} / \mathrm{d} \\
\mathrm{dh} / \mathrm{dl} & =0.0030 \\
\mathrm{n}_{\mathrm{e}} & =0.25
\end{aligned}
$$

A. Determine average linear velocity:

$$
V_{x}=\frac{k \frac{\mathrm{dh}}{\mathrm{dl}}}{\mathrm{n}_{e}}=\frac{\left(7.3 \frac{\mathrm{~m}}{\mathrm{~d}}\right)(0.0030)}{0.25}
$$

$$
\mathrm{v}_{\mathrm{x}}=0.0876 \mathrm{~m} / \mathrm{d}
$$

B. Determine longitudinal dispersion coefficient:

From Equation 11-9:

$$
\begin{aligned}
a_{L} & =0.00175 \mathrm{~L}^{1.46} \\
a_{L} & =0.0175(25 \mathrm{~m})^{1.46} \\
a_{L} & =1.92 \mathrm{~m} \\
D_{L} & =a_{L} v_{X}+D^{*} \quad\left(\text { assume } D^{*}=0\right) \\
D_{L} & =(1.92 \mathrm{~m})(0.0876 \mathrm{~m} / \mathrm{d}) \\
D_{L} & =0.168 \mathrm{~m}^{2} / \mathrm{d} \\
\text { and }: v_{X} & =0.0876 \mathrm{~m} / \mathrm{d} \\
L & =25 \mathrm{~m} \\
t & =300 \mathrm{~d} \\
C_{0} & =1250 \mathrm{mg} / \mathrm{L}
\end{aligned}
$$

C. Using Equation 11-8:

$$
\begin{aligned}
& C=\frac{1250 \mathrm{mg} / \mathrm{L}}{2}\left[\operatorname{erfc}\left(\frac{25 \mathrm{~m}-(0.0876 \mathrm{~m} / \mathrm{d} \times 300 \mathrm{~d})}{2 \times \sqrt{0.168 \frac{\mathrm{~m}^{2}}{\mathrm{~d}} \times 300 \mathrm{~d}}}\right)+\exp \left(\frac{0.0876 \mathrm{~m} / \mathrm{d} \times 25 \mathrm{~m}}{0.168 \frac{\mathrm{~m}^{2}}{\mathrm{~d}}}\right)\right. \\
& \left.x \operatorname{erfc}\left(\frac{25 m+(0.0876 \mathrm{~m} / \mathrm{d} \times 300 \mathrm{~d})}{2 \times \sqrt{0.168 \frac{\mathrm{~m}^{2}}{d} \times 300 \mathrm{~d}}}\right)\right] \\
& C=625 \frac{\mathrm{mg}}{\mathrm{~L}}[\operatorname{erfc}(-0.090)+\exp (13.04) \mathrm{xerfc}(3.61)] \\
& \text { Since erfc }(3.61) \simeq 0 \text {, ignore second term. } \\
& C=625 \mathrm{mg} / \mathrm{L}[\operatorname{erfc}(-0.090)] \text { and } \operatorname{erfc}(-x)=1+\operatorname{erf}(x) \\
& \text { From Appendix 13: erf }(0.090)=0.10 \text {, therefore } \\
& \operatorname{erfc}(-0.090)=1+0.10=1.10 \\
& C=625 \mathrm{mg} / \mathrm{L} \times 1.10 \\
& c=690 \mathrm{mg} / \mathrm{L}
\end{aligned}
$$

## Example 2.31

What would the concentration of sodium be at the same time at a monitoring well located 32 m downgradient of the leaking landfill?

Solution:
Given the same landfill with same flow situation and $\mathrm{L}=32 \mathrm{~m}:$

$$
\begin{aligned}
& a_{L}=0.0175(32)^{1.46}=2.76 \mathrm{~m} \\
& D_{L}=a_{L} v_{x}=2.76 \mathrm{~m} \times 0.0876 \mathrm{~m} / \mathrm{d} \\
& D_{\mathrm{L}}=0.241 \mathrm{~m}^{2} / \mathrm{d}
\end{aligned}
$$

Using Equation 11-8:

$$
\begin{aligned}
& C=\frac{1250 \mathrm{mg} / \mathrm{L}}{2}\left[\operatorname{erfc}\left(\frac{32 \mathrm{~m}-(0.00876 \mathrm{~m} / \mathrm{d} \times 300 \mathrm{~d})}{2 \times \sqrt{0.241 \frac{\mathrm{~m}^{2}}{\mathrm{~d}} \times 300 \mathrm{~d}}}\right)+\exp \left(\frac{0.0876 \mathrm{~m} / \mathrm{d} \times 32 \mathrm{~m}}{0.241 \frac{\mathrm{~m}^{2}}{\mathrm{~d}}}\right)\right. \\
& \left.x \operatorname{erfc}\left(\frac{32 \mathrm{~m}+(0.0876 \mathrm{~m} / \mathrm{d} \times 300 \mathrm{~d})}{2 \times \sqrt{0.241 \frac{\mathrm{~m}^{2}}{\mathrm{~d}} \times 300 \mathrm{~d}}}\right)\right] \\
& C=625 \frac{\mathrm{mg}}{\mathrm{~L}}[\operatorname{erfc}(0.336)+\exp (11.6) \mathrm{xerfc}(3.427)]
\end{aligned}
$$

since erfc $(3.427) \simeq 0$, ignore second term.
$C=625 \mathrm{mg} / \mathrm{L}[\operatorname{erfc}(0.336)]$
From Appendix 11, erfc $(0.336) \simeq 0.365$
$C=625 \mathrm{mg} / \mathrm{L}(0.365)$
$C=228 \mathrm{mg} / \mathrm{L}$

